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Gravitational wave emission at high density and strong coupling

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Gracias, gràcies.

Seguir un sol camí és tornar enrere.

Igor Stravinsky

Chapter 1

Introduction

The fact that we are able to detect gravitational waves (GWs) since recently [1] largely broadens our observational ambition on the universe. GWs carry information about compact objects in the universe and also about cosmological parameters that would ideally give information about the early universe (prior to the recombination encoded on the cosmic microwave background). Indeed, prior to recombination, matter was ionized in a plasma state at a certain temperature, and such a matter state emits gravitational radiation, as we will see. Since we receive no electromagnetic radiation from prior to recombination, it is a big deal to study the only information that could possibly be travelling towards us from the very early universe. However, the fact that those GWs would reach us from that long ago traduces into a low frequency radiation (too low to be detected) due to gravitational redshift [2]. Moreover, there are other sources of thermal GWs that reach us in the frequency range and hence blur the spectrum [3].

An example of a compact object that can emit gravitational radiation is found at the cores of neutron stars, where very high densities exist. The case we are going to focus on is can be applied to such objects. However, neutron stars emit thermal GWs at frequencies that are much higher than the ones we are able to detect [4].

Such plasma state is of course formed by strongly-coupled particles, and in such coupling regime computational issues appear. The aim of this work is to guide the reader through a comprehensive road leading to the computation of a correlation function on a strongly-coupled thermal field theory (henceforth denoted as ThFT). This correlator will almost directly give the emission rate of GWs from a thermal source. If we mean to be self-consistent, this is not an easy nor a short road, and requires a lot of exploration of all the topics that merge together to allow the computation of an strongly-coupled correlator. This work has been greatly inspired on a similar paper by L. Castells-Tiestos [2], but with the addition of finite density for the emitting body.

The beginning of the road is a sufficiently deep exploration of the duality between periodic imaginary time quantum mechanics (QM) and quantum statistical mechanics (QSM). We will see that we can compute quantities in QSM through a path integral approach to QM. Statistical physics naturally lead to the first appearance of the star of the show: finite temperature. Once this temperature is identified with imaginary time, we will consider real time in this same formalism to find that we can define causal propagators on a thermal theory. Chapter 2 of the book on thermal field theory by Le Bellac complements the given information very well [13].

To get to a quantum field theory (QFT from now on) from QM one just adds spatial dimensions and specifies an harmonic potential. Thus, the QSM/QM duality can be extended to describe quantum fields at finite temperature, and also at finite chemical potential. From here, a coupling for an spin-2 graviton (completely analog to a GW) with an energy-momentum tensor in a field theory can be derived. This is logical since GWs are sourced by stress-energy tensors.

The second part of the work begins with a review of the celebrated AdS/CFT (AdS stands for anti de-Sitter spacetime and CFT for conformal field theory) conjecture [5] with a computational goal. This means that we will loosely cover the most intuitive ideas behind the duality so that the reader may sense the connection between quantum, strongly coupled conformal field theories and classical field theories living in anti-de Sitter spacetime. In particular, we will relate the imaginary time QFT developed through the first part of the work to an AdS+black hole (BH) theory, where the Hawking temperature is dual to the temperature of the field theory. The chemical potential of a charged BH is also found to be dual to a $U(1)$ current on the QFT side. This part of the work is covered with brilliant accuracy on the book about the gauge/gravity duality by M. Ammon and J. Erdmenger [15].

Once this is done, we will compute at strong coupling the the two-point function for the source of GWs, the stress-energy tensor and we will study its high density limit.

Throughout the work we will use the Minkowski metric as $\eta = \text{diag} = (-,+,+,+)$. The indices in capital letters $M, N, \ldots$ refer to five-dimensional coordinates or higher, the Greek letters $\mu, \nu, \ldots$ refer to four-dimensional coordinates, and the indices $i, j, \ldots$ refer to purely spatial components.

Chapter 2

Quantum Statistical Mechanics

Our story begins by connecting Minkowski and Euclidean metrics. This is easily done if one considers the continuation $t \to -i\tau$, as it yields $t^2 - \mathbf{x}^2 \to -(\tau^2 + \mathbf{x}^2)$, which is an Euclidean metric up to a sign. This is called Wick rotation, and it is just a fancy way of saying that time is now purely imaginary, given that $\tau \in \mathbb{R}$. In the following we will see how this analytic continuation yields a crucial connection in order to study quantum fields at finite temperature. In particular, we will connect quantum statistical mechanics (QSM) with a (1+0)-dimensional quantum field theory (QFT), which is nothing but quantum mechanics (QM).

2.1 Quantum-mechanical path integral in imaginary time

In conventional quantum mechanics (with one spatial direction for simplicity), the probability amplitude associated with finding a particle (subject to a time-independent potential) at position q' and time t' knowing that it was at position q at time t is given by ($\hat{U}(t'-t)$ is a time evolution operator)

$$F(q', t'; q, t) = \langle q'|\hat{U}(t'-t)|q\rangle = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle \,, \tag{2.1}$$

which can be continued to imaginary values of time ($t \to -i\tau$):

$$F(q', -i\tau'; q, -i\tau) = \langle q'|\hat{U}(-i(\tau'-\tau))|q\rangle = \langle q'|e^{-\hat{H}(\tau'-\tau)}|q\rangle \,. \tag{2.2}$$

Now, the key to translate this quantity to a path integral comes from noticing that the imaginary-time evolution operator can be understood as a concatenation of operators in the style of

$$\hat{U}(-i\tau', -i\tau) = \hat{U}(-i\tau_{n+1}, -i\tau_n)\hat{U}(-i\tau_n, -i\tau_{n-1})...\hat{U}(-i\tau_1, -i\tau_0) \tag{2.3}$$

with $\tau_{n+1} = \tau'$ and $\tau_0 = \tau$. Now, if we take $\epsilon = (\tau_{j+1} - \tau_j) = \frac{(\tau'-\tau)}{(n+1)} \to 0$ (that is, if we take infinite time subdivisions), we can write the infinitesimal time evolution operators and insert n identity operators in the style of $\int dq_j \left|q_j\right\rangle \left\langle q_j\right| = 1$ at times $\tau_1, ..., \tau_n$ between each time evolution operator to find that

$$\hat{U}(-i\tau', -i\tau) = \lim_{\epsilon\to 0} \int dq_1...dq_n \hat{U}(-i\epsilon) \left|q_n\right\rangle \left\langle q_n\right| ...\hat{U}(-i\epsilon) \left|q_1\right\rangle \left\langle q_1\right| \hat{U}(-i\epsilon), \tag{2.4}$$

which finally yields

$$F(q', -i\tau'; q, -i\tau) = \lim_{\epsilon\to 0} \int dq_1...dq_n \left\langle q_{n+1}\right| \hat{U}(-i\epsilon) \left|q_n\right\rangle ... \left\langle q_2\right| \hat{U}(-i\epsilon) \left|q_1\right\rangle \left\langle q_1\right| \hat{U}(-i\epsilon) \left|q_0\right\rangle . \tag{2.5}$$

Now we can write the explicit Hamiltonian and work on the matrix elements separately, giving

$$\left\langle q_{j+1}\right| \hat{U}(-i\epsilon) \left|q_j\right\rangle \approx \int dp_j \left\langle q_{j+1}\right| e^{-\epsilon\frac{\hat{p}^2}{2m}} \left|p_j\right\rangle \left\langle p_j\right| e^{-\epsilon V(\hat{q})} \left|q_j\right\rangle . \tag{2.6}$$

Using that $\langle q_i|p_j\rangle = \frac{1}{\sqrt{2\pi}} e^{iq_ip_j}$ and performing Gaussian integration in (2.6) one gets

$$\left\langle q_{j+1}\right| \hat{U}(-i\epsilon) \left|q_j\right\rangle = \sqrt{\frac{m}{2\pi\epsilon}} e^{-m(q_{j+1}-q_j)^2/2\epsilon} e^{-\epsilon V(q_j)}, \tag{2.7}$$

which introduced in (2.5) finally yields

$$F(q', -i\tau'; q, -i\tau) = \lim_{\epsilon\to 0} \left(\frac{m}{2\pi\epsilon}\right)^{1/2} \int \prod_{j=1}^{n} \left[\left(\frac{m}{2\pi\epsilon}\right)^{1/2} dq_j\right] e^{-\epsilon\sum_{j=0}^{n} s(q_{j+1}, q_j)}, \tag{2.8}$$

with $s(q_{j+1}, q_j) = \frac{m(q_{j+1}-q_j)^2}{2\epsilon^2} + V(q_j) \sim \frac{1}{2}m\dot{q}(\tau_j) + V(q(\tau_j))$. Now we clearly see that the quantity in the exponent reduces to

$$\epsilon \sum_{j=0}^{n} s(q_{j+1}, q_j) = \int_{\tau}^{\tau'} d\tau'' \left[\frac{1}{2}m\dot{q}^2(\tau'') + V(q(\tau''))\right] = S_E(\tau' - \tau), \tag{2.9}$$

which is finally our long awaited Euclidean action. The integration measure is formally summed up as $\mathscr{D}q(\tau'')$, so that the probability amplitude can be cast in the form of a path integral:

$$F(q', -i\tau'; q, -i\tau) = \int \mathscr{D}q(\tau'') e^{-S_E(\tau'-\tau)} \tag{2.10}$$

with boundary conditions $q(\tau) = q$, $q(\tau') = q'$.

As a final comment, since the action coming from the Hamiltonian with real, ordinary time is $S(t) = \int_0^t dt' \left[\frac{1}{2}m\dot{q}^2(t') - V(q(t'))\right]$, one can easily derive that

$$\begin{aligned} S(-i\tau) &= \int_0^{-i\tau} dt' \left[\frac{1}{2}m\dot{q}^2(t') - V(q(t'))\right] \\ &= \int_0^{\tau} d(-i\tau') \left[-\frac{1}{2}m\dot{q}^2(-i\tau') - V(q(-i\tau'))\right], \end{aligned} \tag{2.11}$$

which gives the relation

$$S(-i\tau) = iS_E(\tau). \tag{2.12}$$

2.2 Connection with quantum statistical mechanics

Let us briefly recall the basis of QSM. According to the laws of quantum mechanics, a given system will be in an state $|\psi_i\rangle$ with probability p_i. Given this situation, we can write the expectation value of an operator as

$$\langle \hat{A} \rangle = \sum_i p_i \langle \psi_i | \hat{A} | \psi_i \rangle = \sum_{i,j} p_i \langle \psi_i | \psi_j \rangle \langle \psi_j | \hat{A} | \psi_i \rangle = \mathrm{Tr}\left\{ \left(\sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \hat{A} \right\}, \tag{2.13}$$

where in the last step we just commuted the two factors to obtain a trace. The term inside the parenthesis will define the density operator $\hat{\rho}$ which describes the system. In a canonical ensemble (in which the number of particles is fixed), the probability of occupation of the i-th energy level (corresponding to a state $|\psi_i\rangle$ which satisfies $\hat{H}|\psi_i\rangle = E_i |\psi_i\rangle$) is given by $p_i = Ne^{-\beta E_i}$, with $\beta = T^{-1}$ ($k_B = 1$, T is the temperature) and N a normalization factor. The normalization factor is obtained by observing that, if we set $\hat{A}$ to be the identity operator, $\hat{A} = \hat{I}$, the condition $\mathrm{Tr}\{\hat{\rho}\} = \sum_i p_i = 1$ arises, yielding

$$\sum_i p_i = N \sum_i e^{-\beta E_i} = 1 \to Z^{-1}(\beta) \equiv N = \frac{1}{\sum_i e^{-\beta E_i}}. \tag{2.14}$$

In (2.14) we have defined $Z(\beta)$, which is the **partition function** of the system. Now we can use the fact that $\{|\psi_i\rangle\}$ is a complete basis to rewrite the partition function as

$$Z(\beta) = \sum_j \langle \psi_j | e^{-\beta \hat{H}} |\psi_j\rangle = \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \right\}. \tag{2.15}$$

Now we can write the density operator as

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i| = \frac{e^{-\beta \hat{H}} \sum_i |\psi_i\rangle \langle \psi_i|}{\sum_j \langle \psi_j | e^{-\beta \hat{H}} | \psi_j \rangle} = \frac{e^{-\beta \hat{H}}}{Z(\beta)}, \tag{2.16}$$

which now allows writing the thermal average of an operator as

$$\langle \hat{A} \rangle_\beta = \frac{1}{Z(\beta)} \operatorname{Tr}\left\{ e^{-\beta \hat{H}} \hat{A} \right\}. \tag{2.17}$$

The trick to connect with (2.10) is to consider $e^{-\beta \hat{H}} = \hat{U}(-i\beta)$ as an evolution operator in imaginary time, so that we can write

$$Z(\beta) = \operatorname{Tr}\left\{ e^{-\beta \hat{H}} \right\} = \int dq \, \langle q | e^{-\beta \hat{H}} | q \rangle = \int dq F(q, -i\beta; q, 0). \tag{2.18}$$

Putting the dq inside the measure $\mathscr{D}q(\tau'')$ yields

$$Z(\beta) = \int \mathscr{D}q(\tau) e^{-\int_0^\beta d\tau \left(\frac{1}{2} m \dot{q}^2(\tau) + V(q(\tau)) \right)} = \int \mathscr{D}q(\tau) e^{-S_E(\beta)} \tag{2.19}$$

with the boundary condition $q(\beta) = q(0)$. Now, (2.19) reveals a deep connection: ordinary quantum mechanics with β-periodic imaginary time is equivalent to a system in quantum statistical mechanics at $T = \beta^{-1}$. This is in fact the simplest case of a more general statement, given that QM is nothing but a $(1+0)$-dimensional QFT. Generally, a d-dimensional Euclidean QFT with periodic boundary conditions in time is equivalent to QSM in d-spacetime dimensions[1]. We will clearly see this in the next chapter, but for now let us develop the basics of this connection with no spatial dimensions.

As it is done in every QFT, we can define a generating functional in order to obtain the correlation functions of the theory:

$$Z(\beta; j) = \int \mathscr{D}q(\tau) e^{-S_E(\beta) + \int_0^\beta d\tau \, j(\tau) q(\tau)}. \tag{2.20}$$

The correlation functions are obtained through functional differentiation. For example, we can extract the propagator in imaginary time from

$$\frac{1}{Z(\beta)} \frac{\delta^2 Z(\beta; j)}{\delta j(\tau_1) \delta j(\tau_2)} \bigg|_{j=0} = \frac{1}{Z(\beta)} \int \mathscr{D}q(\tau) q(\tau_1) q(\tau_2) e^{-S_E(\beta)}. \tag{2.21}$$

The generating functional (2.20) represents the QFT point of view, while the thermal average (2.17) represents the QSM point of view. In fact, one can show that if $\hat{A} = T[\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)]$,

[1] This is enough for us, although if one was to consider fermionic behaviour encoded in anticommuting position operators (Grassman variables), boundary conditions emerge antiperiodic $q(\beta) = -q(0)$.

(2.17) and (2.20) are equivalent. The time ordering in imaginary time we used to define the operator is

$$T[\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)] = \theta(\tau_1-\tau_2)\hat{q}(-i\tau_1)\hat{q}(-i\tau_2) + \theta(\tau_2-\tau_1)\hat{q}(-i\tau_2)\hat{q}(-i\tau_1). \tag{2.22}$$

If we write $\hat{q}(-i\tau) = e^{\hat{H}\tau}\hat{q}e^{-\hat{H}\tau}$, with $\hat{q} = \hat{q}(0)$ the position operator in the Schrödinger picture, it is pretty straightforward to show that (2.17) and (2.20) are saying the same thing, given that (for $\tau_1 > \tau_2$)

$$\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\hat{q}(-i\tau_1)\hat{q}(-i\tau_2)\Big\} = \int dq dq_1 dq_2 q_1 q_2 \langle q|e^{-(\beta-\tau_1)\hat{H}}|q_2\rangle \langle q_2|e^{-(\tau_1-\tau_2)\hat{H}}|q_1\rangle \langle q_1|e^{-\tau_2\hat{H}}|q\rangle\,, \tag{2.23}$$

which is the beginning of the procedure we followed to obtain (2.10). All in all, we can state that the thermal average of two position operators at (imaginary) times τ_1, τ_2 gives the propagator of the theory. The time ordering appears naturally because the positions commute inside of the path integrals, and the operators effectively do so inside of the $T[\dots]$.

We can define the propagator in imaginary time in a handier way as

$$\Delta(\tau-\tau') = \langle T[\hat{q}(-i\tau)\hat{q}(-i\tau')]\rangle_\beta\,, \tag{2.24}$$

which satisfies this periodicity condition in imaginary time

$$\Delta(\tau-\beta) = \Delta(\tau). \tag{2.25}$$

One proves (2.25) almost directly using the periodicity of the trace and writing $\hat{q}(-i\beta) = e^{\hat{H}\beta}\hat{q}(0)e^{-\hat{H}\beta}$. Bear in mind that up to now $\tau \in [0,\beta]$. We can Fourier transform the imaginary time propagator $\Delta(\tau)$ taking into account the periodicity condition (2.25), that is

$$\Delta(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau}\Delta(\tau), \tag{2.26}$$

with the inverse

$$\Delta(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\Delta(i\omega_n), \tag{2.27}$$

and the discrete frequencies

$$\omega_n = \frac{2\pi n}{\beta}, \tag{2.28}$$

which are called **Matsubara frequencies**. The imaginary time propagator is usually referred to as the **Matsubara propagator**.

2.3 The harmonic oscillator

In order to make the connection with a free field, we introduce the harmonic oscillator potential (for a particle of mass $m = 1$)

$$V(q) = \frac{1}{2}\omega^2 q^2. \tag{2.29}$$

Now we can write the Euclidean action as

$$S_E(\beta) = \int_0^\beta d\tau \left[\frac{1}{2}\dot{q}^2(\tau) + \frac{1}{2}\omega^2 q^2(\tau)\right], \tag{2.30}$$

where the first term can be integrated by parts as

$$\int_0^\beta d\tau \dot{q}^2(\tau) = \int_0^\beta d\tau \left[\frac{d}{d\tau}\left(\dot{q}(\tau)q(\tau)\right) - \ddot{q}(\tau)q(\tau)\right] = -\int_0^\beta d\tau \ddot{q}(\tau)q(\tau) \tag{2.31}$$

because the first term cancels (this is a crucial detail in order to perform Gaussian integration) due to $q(0) = q(\beta)$. This allows writing the generating functional as

$$\begin{aligned} Z(\beta : j) = \int \mathscr{D}q(\tau) \exp&\left\{-\int d\tau' \int_0^\beta d\tau \frac{1}{2} q(\tau)\left[\delta(\tau - \tau')\left(-\frac{d^2}{d\tau'^2} + \omega^2\right)\right] q(\tau')\right\} \\ \times \exp&\left\{\int d\tau' \int_0^\beta d\tau j(\tau) q(\tau') \delta(\tau - \tau')\right\}. \end{aligned} \tag{2.32}$$

Fortunately, (2.32) is now a Gaussian integral that can be easily computed considering that

$$\begin{aligned} \int dx_1 ... dx_n \exp\left\{-\frac{1}{2}x^T A x + J^T x\right\} &= \int dx_1 ... dx_n \exp\left\{-\frac{1}{2}x_i \delta_{ij} A_{jj} x_j + J_i x_j \delta_{ij}\right\} \\ &= C \exp\left\{\frac{1}{2} J^T A^{-1} J\right\}, \end{aligned} \tag{2.33}$$

where C does not depend on J. Note that in our case A^{-1} is the inverse of $\left[\delta(\tau - \tau')\left(-\frac{d^2}{d\tau'^2} + \omega^2\right)\right]$ in the sense of the operators, so that our A^{-1} is really a function $A^{-1}(\tau, \tau') \equiv K(\tau, \tau')$ defined by

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right) K(\tau, \tau') = \delta(\tau - \tau'), \tag{2.34}$$

while the final form of the generating functional is given by

$$Z(\beta; j) = Z(\beta) \exp\left\{\frac{1}{2}\int\int d\tau d\tau' j(\tau) K(\tau, \tau') j(\tau')\right\}. \tag{2.35}$$

Note that (2.34) tells us that $K(\tau, \tau')$ is a Green function for the equation of motion of an harmonic oscillator in imaginary time (if $\tau \to it$, we recover the usual equation for an harmonic oscillator). The interpretation of $K(\tau,\tau')$ as a propagator could not come in a more natural way.

If one plugs (2.35) into (2.20) it is immediate to notice that $K(\tau,\tau') = \Delta_F(\tau-\tau')$ is a propagator in imaginary time, where the F (*Free*) is a consequence of our choice of potential. Now one happily solves (2.34) with the periodic boundary condition $\Delta(\tau) = \Delta(\tau-\beta)$ and $\tau \in [0,\beta]$ *a priori*. An strategy to solve (2.34) is to separate the problem into two regions, separated by the δ barrier in $\tau = 0$, allowing $\tau \in [-\beta,\beta]$. It turns out that the solution is perfectly fine for negative values of τ and it is given by

$$\Delta_F(\tau) = \frac{1}{2\omega}\left[(1+n(\omega))e^{-\omega\tau} + n(\omega)e^{\omega\tau}\right], \tag{2.36}$$

with $n(\omega)$ the Bose-Einstein distribution

$$n(\omega) = \frac{1}{e^{\beta|\omega|}-1}. \tag{2.37}$$

2.4 What if we let time to be complex?

Now we want to let time to be complex ($t \in \mathbb{C}$) rather than purely imaginary. The first limitation that comes to mind is time ordering. We will need new criteria for ordering complex times. This criteria cannot be the obvious ones of ordering times attending only to their real/imaginary parts, but we will rather need a "sense" of ordering: a path in complex time. This is a beautiful but rather technical way to face the problem and we will not consider here. It is known as the real-time formalism. If the reader is interested, Le Bellac chapter 3 [13] delivers a very enlightening discussion.

From now on we will work with $t \in \mathbb{C}\ /\ t = \mathrm{Re}\{t\} + i\,\mathrm{Im}\{t\}$. On this basis we can define the following thermal correlation functions

$$\begin{aligned} D^{>}(t,t') &= \langle \hat{q}(t)\hat{q}(t')\rangle_\beta\,, \\ D^{<}(t,t') &= \langle \hat{q}(t')\hat{q}(t)\rangle_\beta = D^{>}(t',t), \end{aligned} \tag{2.38}$$

which are called **Wightman propagators**. Taking the traces in (2.38) as a sum over Hamiltonian eigenstates $\hat{H}\,|n\rangle = E_n\,|n\rangle$ and inserting a resolution of the identity $\sum_m |m\rangle\,\langle m|$, we can rewrite the expressions as

$$\begin{aligned}
D^{>}(t,t') &= \frac{1}{Z(\beta)}\sum_{n,m} e^{i\,\mathrm{Re}\{t-t'\}(E_n-E_m)}e^{-(\beta+\mathrm{Im}\{t-t'\})E_n}e^{\mathrm{Im}\{t-t'\}E_m} \\
&\quad\times |\langle n|\hat{q}(0)|m\rangle|^2, \\
D^{<}(t,t') &= \frac{1}{Z(\beta)}\sum_{n,m} e^{-i\,\mathrm{Re}\{t-t'\}(E_n-E_m)}e^{-(\beta-\mathrm{Im}\{t-t'\})E_n}e^{-\mathrm{Im}\{t-t'\}E_m} \\
&\quad\times |\langle n|\hat{q}(0)|m\rangle|^2,
\end{aligned} \tag{2.39}$$

which depend solely on time differences $(t-t')$, and hence there exists translation invariance. From (2.39) we can extract the domain in which the correlation functions are well defined, that is, we need to impose boundaries to the imaginary part of the time difference so that the exponentials do not explode for high energies. For $D^{>}(t,t')$ one easily finds that, in order for it to be well defined we need

$$-\beta \leqslant \mathrm{Im}\{t-t'\} \leqslant 0, \tag{2.40}$$

and for $D^{<}(t,t')$ one needs

$$\beta \geqslant \mathrm{Im}\{t-t'\} \geqslant 0. \tag{2.41}$$

(2.40) and (2.41) state the domain of definition of (2.38).

If we write the thermal weight operator as an imaginary time evolution operator for a complex-timed position operator, that is

$$\hat{q}(t+i\beta) = e^{-\beta\hat{H}}\hat{q}(t)e^{\beta\hat{H}}, \tag{2.42}$$

we can derive a periodic property for (2.38) by inserting an identity ($\hat{I} = e^{-\beta\hat{H}}e^{\beta\hat{H}}$), namely

$$\begin{aligned}
D^{>}(t,t') &= \frac{1}{Z(\beta)}\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\hat{q}(t)e^{\beta\hat{H}}e^{-\beta\hat{H}}\hat{q}(t')\Big\} = \frac{1}{Z(\beta)}\mathrm{Tr}\Big\{\hat{q}(t')\hat{q}(t+i\beta)e^{-\beta\hat{H}}\Big\} \\
&= D^{<}(t+i\beta,t').
\end{aligned} \tag{2.43}$$

As we see in (2.43), the periodicity in imaginary time is still encoded there, defining the so-called Kubo-Martin-Schwinger (KMS) relation:

$$D^{>}(t,t') = D^{<}(t+i\beta,t'), \tag{2.44}$$

which is the extension of the periodicity property of the Matsubara (2.25) propagator for complex times. Indeed, (2.44) for purely imaginary times (take $t'=0$, $\mathrm{Re}\{t\}=0$, $\tau = -\mathrm{Im}\{t\}$) reads

$$D^{>}(-i\tau,0) = D^{<}(-i(\tau-\beta),0), \tag{2.45}$$

which is nothing but the Matsubara propagator (2.25), namely

$$D^{>}(-i\tau, 0) = \Delta(\tau) \tag{2.46}$$

for $\tau \in [0, \beta]$. As an aside note meant for later reference, we write the purely real time-ordered propagator, which is of course the usual

$$D(t, t') = \langle T\left[\hat{q}(t), \hat{q}(t')\right]\rangle_{\beta} = \theta(t - t')D^{>}(t, t') + \theta(t' - t)D^{<}(t, t'). \tag{2.47}$$

2.5 The spectral function

Now let us consider the Fourier transforms of the Wightman propagators in (2.38). Since they only depend on time differences, we will use a handier notation by defining $D^{>(<)}(t, 0) = D^{>(<)}(t)$. The Fourier transform of both quantities is defined as (recall that $t \in \mathbb{C}$)

$$\begin{aligned} D^{>}(k_0) &= \int dt e^{ik_0 t} D^{>}(t), \\ D^{<}(k_0) &= \int dt e^{ik_0 t} D^{<}(t) = \int dt e^{ik_0 t} D^{>}(t - i\beta). \end{aligned} \tag{2.48}$$

The expressions in (2.48) for $t \in \mathbb{C}$ would have to be integrated along a complex path C covering $\mathrm{Re}\{t\} \in]-\infty, +\infty[$ and $\mathrm{Im}\{t\} \in [0, \beta]$, and they would have to take into account the domains of definition (2.40) and (2.41). This is of course easier said than done, and it is based in the imaginary time formalism we cited earlier. Since we are mainly interested in real times, let us tiptoe this detail. Note that the second equation in (2.48) is just the Fourier extension of the KMS relation (2.44).

According to the definitions in (2.38), we have that, for real values of time $t \in \mathbb{R}$ (which are the ones that suit our interests), $D^{>}(t)^* = D^{>}(-t) = D^{<}(t)$, and that this implies the reality of $D^{>(<)}(k_0)$, $D^{>(<)}(k_0)^* = D^{>(<)}(k_0)$. Now that we know this, a simple calculation gives

$$D^{>}(t)^* = D^{<}(t) \to D^{>}(k_0) = D^{<}(-k_0)^* = D^{<}(-k_0) \tag{2.49}$$

and

$$D^{<}(k_0) = \int dt\, e^{ik_0 t} D^{>}(t - i\beta) = e^{-\beta k_0} \int dt\, e^{ik_0(t - i\beta)} D^{>}(t - i\beta) = e^{-\beta k_0} D^{>}(k_0), \tag{2.50}$$

which can be summed up in a compact expression giving the relation

$$D^{<}(k_0) = D^{>}(-k_0) = e^{-\beta k_0} D^{>}(k_0), \tag{2.51}$$

which will prove to be of great importance in the following.

The previous reasoning gives a natural intuition of the following definition,

$$\rho(k_0) = D^>(k_0) - D^<(k_0), \tag{2.52}$$

which is called **spectral function**. The spectral function in (2.52) is in fact nothing but the Fourier transform of the thermal average of the position operators commutator, namely

$$\rho(k_0) = \int dt e^{ik_0 t} \langle [\hat{q}(t), \hat{q}(0)] \rangle_\beta \,. \tag{2.53}$$

The spectral function is an odd function of k_0,

$$\rho(k_0) = -\rho(-k_0). \tag{2.54}$$

From (2.53) we can extract the free spectral function by writing the expansion of the position operator in terms of the creation/annihilation operators ($[\hat{a}, \hat{a}^\dagger] = 1$)

$$\hat{q}(t) = \frac{1}{\sqrt{2\omega}} \left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t} \right), \tag{2.55}$$

which almost directly gives the expression

$$\rho_F(k_0) = 2\pi \varepsilon(k_0) \delta(k_0^2 - \omega^2) \tag{2.56}$$

with $\varepsilon(k_0)$ the sign function. Note that (2.56) does not have any β-dependence, and thus it does not present any thermal characteristics.

In a general theory, we need not to specify the character of the spectral function, and the general result follows from using (2.51) in the definition (2.52), giving

$$\rho(k_0) = D^>(k_0) - D^<(k_0) = D^>(k_0)(1 - e^{-\beta k_0}), \tag{2.57}$$

which directly yields

$$\begin{aligned} D^>(k_0) &= (1 + f(k_0))\rho(k_0), \\ D^<(k_0) &= f(k_0)\rho(k_0); \end{aligned} \tag{2.58}$$

with (note from (2.37) that $n(k_0) = f(|k_0|)$)

$$f(k_0) = (e^{\beta k_0} - 1)^{-1}. \tag{2.59}$$

Needless to say that $k_0 \in \mathbb{R}$.

It is also interesting to study the extremal limits of $f(k_0)$. The low temperature limit $T \to 0$ yields

$$T \to 0 \iff f(k_0) \to 0. \tag{2.60}$$

It is particularly interesting to take the $T \to 0$ limit on the Wightman propagators (2.58). In doing so, one finds that

$$T \to 0 \iff D^<(k_0) \to 0, \tag{2.61}$$

so that in the low temperature limit $\rho(k_0) \to D^>(k_0)$. The classical (low energy) limit $k_0 \ll T$ gives

$$k_0 \ll T \iff f(k_0) \to \frac{T}{k_0} \gg 1. \tag{2.62}$$

2.6 Extending the Matsubara propagator

Around (2.48), we got a flavour of the difficulties that arise when considering complex times in the formalism. We need to order complex times, and so we need a (non-trivial) path to give them a sense of order. Think about how you would order random points inside a circle. A kind of spiral from the center to the border would roughly do the job. That is roughly the idea behind the real-time formalism we named earlier.

However, the Fourier side of the coin provides, as always, a shortcut. We can make use of the quantities we defined in the previous section to extend the Matsubara propagator (2.48) to a continuum of frequencies, abandoning the discrete Matsubara frequencies (2.28) which are a consequence of considering purely imaginary times. Having said this, let us work with the Matsubara propagator in the interval $\tau \in [0, \beta]$, that is,

$$\Delta(\tau) = D^>(-i\tau) = \int \frac{dk_0}{2\pi} e^{-k_0\tau} D^>(k_0), \tag{2.63}$$

where we have used (2.46) and the inverse of (2.48). Now we can use (2.26) and (2.58) in order to write the preceding equation in terms of the spectral function, that is,

$$\begin{aligned} \Delta(i\omega_n) &= \int_0^\beta d\tau \int \frac{dk_0}{2\pi} e^{(i\omega_n - k_0)\tau}(1 + f(k_0))\rho(k_0) \\ &= -\int \frac{dk_0}{2\pi} \frac{\rho(k_0)}{i\omega_n - k_0}, \end{aligned} \tag{2.64}$$

where the integration is simple. If we take the free spectral function (2.56), the k_0-integral is also simple enough thanks to the δ-terms that stem from the decomposition

$$\delta(k_0^2 - \omega^2) = \frac{1}{2\omega} \left[\delta(k_0 - \omega) + \delta(k_0 + \omega)\right]. \tag{2.65}$$

All in all, we can write

$$\Delta_F(i\omega_n) = \frac{1}{\omega_n^2 + \omega^2}, \tag{2.66}$$

which is indeed the Fourier transform of the solution to (2.34).

However, this procedure is clearly limited to purely imaginary times and tells nothing new. Imagine if we allowed $i\omega_n \to -i(iq_0 - \eta) = q_0 + i\eta \equiv z$. In other words, imagine that $i\omega_n$ becomes a continuous complex variable. In doing so, one extends the Matsubara propagator to arbitrary values of complex time (or conversely, energy). If one is interested in purely real values of time (like we do), then η can be sent to 0, as we will see in a moment. In order for this continuation to be analytic, we obviously have to demand analyticity all along the real axis ($\Delta(z)$ is analytic for $\text{Im}\{z\} = 0$), aside from the rather obvious condition of $\Delta(|z| \to \infty) \to 0$ (we cannot recieve information from infinity!). All in all, our demands seem to flow into a single one and only analytic continuation:

$$\Delta(z) = -\int \frac{dk_0}{2\pi} \frac{\rho(k_0)}{z - k_0}, \tag{2.67}$$

which generalizes (2.64). Indeed, (2.67) is the key to define the physically interesting propagators, which we study in the next section.

2.7 A glimpse on linear response theory

Up to this point, we have developed a solid formalism. We have connected Euclidean QFT with QSM by making time imaginary. We have also extended the connection for complex and real times, which is the same that saying that we have allowed the Matsubara frequencies to be complex. We have then got to a point where writing real time causal propagators seems realistic. The physical propagator we are mainly interested in is the retarded propagator. We will get to this quantity and we will relate it to the Wightman propagators through linear response theory, which allows for a very transparent and unambiguous physical interpretation.

There is a key mathematical expression for our purposes at this section, which is worth mentioning beforehand. This identity is a form of the so-called Sokhotski-Plemelj theorem, which states that in the limit $\eta \to 0$ we can write

$$\frac{1}{x \pm i\eta} = \mathbf{P}\frac{1}{x} \mp i\pi\delta(x), \tag{2.68}$$

where $\mathbf{P}\frac{1}{x} = \frac{x}{x^2+\eta^2}$ denotes a principal value.

2.7.1 Classical approach

Let us consider a classical damped and forced harmonic oscillator (for a particle of mass $m = 1$) described by

$$\ddot{q}(t) + \omega^2 q(t) + \eta\dot{q}(t) = j(t), \tag{2.69}$$

which in Fourier space reads

$$\left[-(k_0^2 - \omega^2) - i\eta k_0\right] q(k_0) = j(k_0). \tag{2.70}$$

From (2.70) one can express the solution in terms of a so-called response function, which regulates the effect of the external source, namely,

$$\chi(k_0) = \frac{-1}{(k_0^2 - \omega^2) + i\eta k_0}, \tag{2.71}$$

and hence the solution to (2.69) is given by[2]

$$q(k_0) = \chi(k_0) j(k_0) \tag{2.72}$$

We can decompose (2.71) as

$$\chi(k_0) = -\frac{(k_0^2 - \omega^2)}{(k_0^2 - \omega^2)^2 + \eta^2 k_0^2} + \frac{i\eta k_0}{(k_0^2 - \omega^2)^2 + \eta^2 k_0^2}. \tag{2.73}$$

In order to make a connection, let us take a limit on the viscous term, $\eta \to 0^+$. Now, through (2.68), the imaginary part of (2.73) can be written in a very fancy way as

$$\mathrm{Im}\{\chi(k_0)\} = \varepsilon(k_0)\pi\delta(k_0^2 - \omega^2), \tag{2.74}$$

which gracefully gives

$$\mathrm{Im}\{\chi(k_0)\} = \frac{1}{2}\rho_F(k_0). \tag{2.75}$$

[2]There could be a constant $q(0)$ standing for the initial value.

From (2.75) we can read the interpretation of the spectral function as a measure of dissipation. This is easily seen if one computes the energy dissipated due to the action of the forcing term. This amounts

$$\Delta E(t) = \int dt' \dot{q}(t') j(t'-t) = \int \frac{dk_0}{2\pi} e^{-ik_0 t}(-ik_0)\chi(k_0)|F(k_0)|^2, \tag{2.76}$$

and we see that in order to keep $\Delta E(t)$ real we have to consider the imaginary part of the response function, which is the spectral function in the $\eta \to 0$ limit.

2.7.2 Quantum mechanical approach

In real-time quantum mechanics, a generating functional is built by adding an external source term to the action of the form of $\int dt j(t)q(t)$, which modifies the equation of motion of $q(t)$. In particular, for the harmonic potential the classical equation of motion would be

$$\left(\frac{d^2}{dt^2} + \omega^2\right) q(t) = j(t), \tag{2.77}$$

that is, a forced harmonic oscillator. The action of the source needs time to communicate its presence to the system, and so we expect the retarded propagator to emerge naturally. A very simple computation in quantum-mechanical perturbation theory (you can find it in [15]) gives the deviation from the free expectation value that $q(t)$ acquires from $j(t)$

$$\delta \langle \hat{q}(t) \rangle_\beta = -i \int_{-\infty}^{+\infty} dt' j(t')\theta(t-t') \langle [\hat{q}(t), \hat{q}(t')] \rangle_\beta . \tag{2.78}$$

Hence, the source is in a convolution with a Green function: the retarded propagator

$$D_R(t) = -i \langle \theta(t)[\hat{q}(t), \hat{q}(0)] \rangle_\beta , \tag{2.79}$$

which causally propagates the effects of the source. The advanced propagator follows from inverting the time arrow, yielding

$$D_A(t) = i \langle \theta(-t)[\hat{q}(t), \hat{q}(0)] \rangle_\beta . \tag{2.80}$$

The concept of retarded/advanced only makes sense in real time (which is our frame to say later/earlier), and thus $t \in \mathbb{R}$ in both cases. If we take the retarded propagator defined in (2.79), we can see that, through the Fourier transform of the very definition of the spectral function (2.52)

$$D_R(t) = -i\theta(t)[D^>(t) - D^<(t)] = -i\theta(t) \int \frac{dk_0}{2\pi} e^{-ik_0 t}\rho(k_0). \tag{2.81}$$

In order to make some progress with (2.81), we need another representation of the Heaviside function, namely

$$\theta(t) = i \int \frac{dq_0}{2\pi} \frac{e^{-iq_0 t}}{q_0 + i\eta} \tag{2.82}$$

with $\eta \to 0^+$. The reader may prove (2.82) by looking for a one-pole function that is equal to zero for $t < 0$. Now that we know this, (2.81) can be Fourier transformed to give

$$D_R(k_0) = \int \frac{dq_0}{2\pi} \frac{\rho(q_0)}{k_0 + i\eta - q_0}, \tag{2.83}$$

which can be written in a compact way following (2.67) as

$$D_R(k_0) = -\Delta(k_0 + i\eta). \tag{2.84}$$

An analogous computation leads to the advanced propagator as well,

$$D_A(k_0) = -\Delta(k_0 - i\eta). \tag{2.85}$$

The results we have obtained confirm the physical validity of the analytic continuation performed in (2.67).

Let us remark that since $\rho(k_0) \in \mathbb{R}$ (given that $D^{>(<)}(k_0) \in \mathbb{R}$), the propagators fulfill the following property:

$$D_A(k_0) = D_R(k_0)^*. \tag{2.86}$$

With (2.68) we can rewrite the retarded propagator as

$$D_R(k_0) = \int \frac{dq_0}{2\pi} \mathbf{P} \frac{\rho(k_0)}{k_0 - q_0} - \frac{i}{2}\rho(k_0), \tag{2.87}$$

which combined with (2.86) yields a very enlightening result:

$$\mathrm{Im}\{D_R(k_0)\} = -\,\mathrm{Im}\{D_A(k_0)\} = -\frac{1}{2}\rho(k_0), \tag{2.88}$$

which relates to the classical case (2.75) up to a sign. It is clear that the dissipative character is encoded in the imaginary part of the retarded/advanced propagator. Both the beauty and the importance of (2.88) reside in a detail: dissipation owes its existence to the discontinuity of the analytically extended Matsubara propagator. Namely,

$$\rho(k_0) = -2\,\mathrm{Im}\{D_R(k_0)\} = -\left(\Delta(k_0 + i\eta) - \Delta(k_0 - i\eta)\right) = -\mathrm{Disc}\Delta(k_0). \tag{2.89}$$

By virtue of (2.88), we can also relate the Wightman two-point functions and the retarded/advanced ones. Indeed, in agreement with (2.58), we can write the following important identity:

$$D^{<}(k_0) = -2f(k_0)\,\mathrm{Im}\{D_R(k_0)\}, \tag{2.90}$$

which is easily converted for the other Wightman propagator,

$$D^{>}(k_0) = -2(1 + f(k_0))\,\mathrm{Im}\{D_R(k_0)\}. \tag{2.91}$$

The quotient of (2.90) and (2.91) yields the ratio

$$\frac{D^{>}(k_0)}{D^{<}(k_0)} = 1 + \frac{1}{f(k_0)}, \tag{2.92}$$

which for the $T \to 0$ limit (2.61) implies

$$D^{>}(k_0) = -2\,\mathrm{Im}\{D_R(k_0)\}, \tag{2.93}$$

and for the classical $k_0 \ll T$ limit implies

$$D^{>}(k_0) = D^{<}(k_0) \simeq -\frac{2T}{k_0}\,\mathrm{Im}\{D_R(k_0)\}. \tag{2.94}$$

2.8 The time-ordered propagator

The natural extension of the previous results is obviously to consider the real-time thermal time-ordered propagator. The real-time-ordered propagator is written in (2.47), and its Fourier transform is

$$D(k_0) = \int dte^{ik_0 t}\left[\theta(t)D^{>}(t) + \theta(-t)D^{<}(t)\right]. \tag{2.95}$$

An adequate treatment of (2.95) allows writing the propagator as (recall that $\varepsilon(t) = \theta(t) - \theta(-t)$)

$$\begin{aligned} D(k_0) &= \int dte^{ik_0 t}\left[\frac{1}{2}\varepsilon(t)\left(D^{>}(t) - D^{<}(t)\right) + \frac{1}{2}\left(D^{>}(t) + D^{<}(t)\right)\right] \\ &= \int dte^{ik_0 t}\left[\frac{1}{2}\varepsilon(t)\left\langle[\hat{q}(t), \hat{q}(0)]\right\rangle_\beta + \frac{1}{2}\left\langle\{\hat{q}(t), \hat{q}(0)\}\right\rangle_\beta\right] \\ &\equiv D_{-}(k_0) + D_{+}(k_0). \end{aligned} \tag{2.96}$$

This is a very suggestive way to write the propagator, because it separates the fluctuating term (the $\{\ldots\}$-term), related to the dissipation by the so-called fluctuation-dissipation theorem.

Note that the second term in (2.96) is nothing but a measure of symmetrized fluctuations in a quantum thermal system. In our case, we state the fluctuation dissipation theorem as[3]

$$D_+(k_0) = -\left[1 + 2f(k_0)\right] \operatorname{Im}\{D_R(k_0)\}, \tag{2.97}$$

given that $\operatorname{Im}\{D_R(k_0)\} = -\frac{1}{2}\rho(k_0)$ is the measure of dissipation. Note that (2.97) is just one half of the the sum of the two Wightman propagators in (2.90), (2.91). Furthermore, we can express (2.97) in terms of $n(k_0)$ through a very short computation (separating the $k_0 > 0$ and $k_0 < 0$ cases), yielding

$$D_+(k_0) = -\varepsilon(k_0)\left[1 + 2n(k_0)\right] \operatorname{Im}\{D_R(k_0)\}. \tag{2.98}$$

Using the representation (2.82) and the fact that $\varepsilon(t) = \theta(t) - \theta(-t)$, we can very easily deduce the following representation for the sign function

$$\varepsilon(t) = -i \int \frac{dk_0}{2\pi} \mathbf{P} \frac{2}{k_0}, \tag{2.99}$$

which is very useful for the computation of (2.96). Indeed, the first term in (2.96), $D_-(k_0)$, can be written as

$$D_-(k_0) = \frac{i}{2}\left[D_R(k_0) + D_A(k_0)\right] = i \operatorname{Re}\{D_R(k_0)\}, \tag{2.100}$$

and hence we can write (2.96) in a very simple way combining (2.98) and (2.100) as

$$D(k_0) = i\left(\operatorname{Re}\{D_R(k_0)\} + i\varepsilon(k_0) \operatorname{Im}\{D_R(k_0)\}\right) - 2\varepsilon(k_0)n(k_0) \operatorname{Im}\{D_R(k_0)\}. \tag{2.101}$$

The only thing left to do is to substitute the expressions for the real and imaginary parts we gave in (2.87). Upon some manipulation, one writes (2.101) as

$$D(k_0) = i \int \frac{dq_0}{2\pi} \left[\mathbf{P}\frac{1}{k_0 - q_0} - \varepsilon(q_0) i\pi\delta(k_0 - q_0)\right] \rho(k_0) + \varepsilon(k_0)n(k_0)\rho(k_0), \tag{2.102}$$

and thanks to the identity (2.68), we can finally give the time-ordered propagator for a general theory described by $\rho(k_0)$ and β:

$$D(k_0) = \int dq_0 \frac{i}{k_0 - q_0 + i\varepsilon(q_0)\eta} \rho(q_0) + \varepsilon(k_0)n(k_0)\rho(k_0). \tag{2.103}$$

Note that only the second term in (2.103) is β-dependent.

[3] You can check (2.97) simply by adding $\frac{1}{2}\left[D^>(k_0) + D^<(k_0)\right]$ using (2.90) and (2.91).

The usual procedures in QFT use perturbation theory around a free non-interacting theory. Therefore our main interest resides in the free form of (2.103), which is obtained by substitution of the free spectral function (2.56), yielding

$$D_F(k_0) = \frac{i}{k_0^2 - \omega^2 + i\eta} + 2\pi n(k_0)\delta(k_0^2 - \omega^2), \tag{2.104}$$

which correctly reduces to the Feynman propagator for $T = 0$.

2.9 The self-energy

The main object in the following discussion will be the self-energy. The self energy can be defined as a perturbative interactive correction to the free propagator like

$$\Delta(z) = \Delta_F(z) + \Delta_F(z)\,[-\Pi(z)]\,\Delta_F(z) + \dots, \tag{2.105}$$

where $\Delta(z)$ is the propagator in the interactive theory. Assuming weak coupling, the series can be resummed to give

$$\Delta^{-1}(z) = \Delta_F^{-1}(z) + \Pi(z). \tag{2.106}$$

The self energy corrects the free propagation by adding “self-interactions” (diagramatically, bubbles) over the course of the propagation from one point in space-time to another.

Using the expression (2.56) for the free spectral function and the expression (2.84) for the retarded propagator $D_R(k_0) = \Delta(k_0 + i\eta)$, we can introduce the retarded self-energy as $D_R(k_0) = D_R^F(k_0) + D_R^F(k_0)\,[-\Pi_R(k_0)]\,D_R^F(k_0) + \dots$, or more explicitly

$$D_R(k_0) = \frac{1}{k_0^2 - (\omega^2 + \mathrm{Re}\{\Pi_R(k_0)\}) + i\,\mathrm{Im}\{\Pi_R(k_0)\}}, \tag{2.107}$$

where we have separated the self-energy into its real and imaginary part and neglected the $i\eta$ term in the denominator. We can see that $\mathrm{Re}\{\Pi_R(k_0)\}$ is just an energy correction, while $\mathrm{Im}\{\Pi_R(k_0)\}$ plays the role of $i\eta$. By taking a look at (2.68) and (2.88), we can see that $\mathrm{Im}\{D_R^F(k_0)\} = \varepsilon(k_0)\pi\delta(k_0^2 - \omega^2) = \frac{1}{2}\rho_F(k_0)$ amounts to two sharp dissipative peaks. Hence, the presence of $\mathrm{Im}\{\Pi_R(k_0)\} >> \eta$ suggests that the peak opens up and becomes a wider resonance. The fact that this happens gives an heuristic understanding of why imaginary parts of self-energies are related with **decay rates**.

Indeed, since the self-energy is a correction to the free propagation (in which the initial and final states are the same ones), one may interpret the self-energy as sum of all probability amplitudes that a one-particle state i evolves into a multi-particle state X to then return to the initial state $f = i$, and hence $-\Pi = \mathscr{M}(i \to i) = \sum_X \mathscr{M}(i \to X)\mathscr{M}(X \to i) = \sum_X |\mathscr{M}(i \to X)|^2$, where the last step is implied by unitarity and we have left out unimportant phase space factors. In views of this interpretation, we can use the **optical theorem** to write

$$\frac{1}{2i}\left[\mathscr{M}(i \to i) - \mathscr{M}^*(i \to i)\right] = \mathrm{Im}\{\mathscr{M}(i \to i)\} = m_i\left[\Gamma^> - \Gamma^<\right] \equiv \Gamma_t, \tag{2.108}$$

where m_i is the mass of the particle (which we previously fixed to unity) and $\Gamma^>$, $\Gamma^<$ are respectively the decay and creation rates of one particle state i. Namely $\Gamma^> = \sum_X \Gamma(i \to X)$, $\Gamma^< = \sum_X \Gamma(X \to i)$. The presence of their difference ensures that Γ_t is a net decay rate, or namely an emission or production rate of i-state particles.

It is not difficult to convince oneself of the relation between the retarded and advanced self-energies

$$\Pi_R(k_0)^* = \Pi_A(k_0), \tag{2.109}$$

which allows us to write a central expression for our purposes:

$$\mathrm{Im}\{\Pi_R(k_0)\} = \frac{1}{2i}\mathrm{Disc}\Pi(k_0 + i\eta) = \frac{1}{2i}\left[\Pi_R(k_0) - \Pi_A(k_0)\right] = \Gamma^> - \Gamma^<. \tag{2.110}$$

The interpretation of (2.110) is pristine: the self-energy bubble can be "cut" to give separately the decay and creation rates of the intermediate states of a given theory. Note the similarity between (2.110) and (2.89).

The natural extension is to consider the time-ordered self-energy just as we did with the propagator. Even though very interesting and instructive, this generalisation is not smooth nor easy to explain because of the appearance of off-diagonal products mixing the real and imaginary parts of the propagators. The most consistent approach is to work in the so-called real time formalism, in which a complex time-ordering path is used to derive a matrix propagator for the theory, which is then diagonalised through a Bogoliubov transformation. For our purposes, this complications are (fortunately) avoidable, since the self-energy diagrams we are going to compute in the field theory are indeed Wightman correlators. If the reader is interested, useful discussion can be found on Le Bellac chapter 3 [13].

Chapter 3

Thermal Field Theory

Now that we have developed a solid formalism with no spatial dimensions, the natural continuation is to introduce (in our case three) spatial degrees of freedom. For the moment, we will deal with the generalisation in the frame of scalar field theory, for which we just have to generalise the time derivatives in the Lagrangian to 4-derivatives including the spatial dimensions.

3.1 The scalar field

Let us introduce a scalar field through a Lagrangian density in Minkowski space, that is

$$S(t) = \int_0^t d^4x \left[\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2 - \mathscr{V}(\varphi) \right], \tag{3.1}$$

with $\varphi(x)$ a real scalar field. Inspired by (2.12), we may write the corresponding Euclidean action as

$$S_E(\beta) = \int_0^\beta d^4x \left[\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \mathscr{V}(\varphi) \right], \tag{3.2}$$

where $(\partial_\mu \varphi)^2 = (\partial_\tau \varphi)^2 + (\nabla \varphi)^2$ and $\int_0^\beta d^4x = \int_0^\beta d\tau \int d^3x$. A straightforward generalization of (2.20) gives a path integral representation of the partition function, namely

$$Z(\beta : j) = \int \mathscr{D}\varphi \exp\left\{ -S_E(\beta) + \int_0^\beta d^4x j(x) \varphi(x) \right\}. \tag{3.3}$$

In the case of the free field ($\mathscr{V}(\varphi) = 0$), we can integrate (3.2) by parts in order to obtain the spatially-generalised free propagator, which responds to the following differential equation in imaginary time

$$-\left(\partial_\tau^2 + \nabla^2 - m^2\right)\Delta_F(x-y) = \delta^4(x-y) \tag{3.4}$$

and can be used to write the free generating functional

$$Z_F(\beta;j) = Z_F(\beta)\exp\left\{\frac{1}{2}\int d^4x \int_0^\beta d^4y j(x)\Delta_F(x-y)j(y)\right\}. \tag{3.5}$$

The Fourier representation of (3.4) gives a direct generalisation of (2.66), where one simply replaces $\omega^2 \to \mathbf{k}^2 + m^2 \equiv \omega_k^2$ so that the Matsubara propagator is written as

$$\Delta_F(i\omega_n, k) = \frac{1}{\omega_n^2 + \omega_k^2}. \tag{3.6}$$

Now that we know how to write the free propagator for a free theory in imaginary time, we can compute Feynman diagrams of interacting theories perturbatively and derive the Feynman rules. This is a simple procedure and the only change with respect to the $T = 0$ Feynman rules will of course be the discretization of the integral measure concerning the Matsubara frequencies. We will skip this since there is no real mistery to it and it does not really interest us. Every other quantity we are interested in generalises directly with the simple rules listed above.

Also, as an aside note, one could also describe fermionic behaviour by changing the Bose-Einstein distribution for a Fermi-Dirac one. Since we will not use fermions, the curious reader is referred to Le Bellac chapter 5 [13] for a thorough description of thermal fermions.

3.2 Local operators

Note that the Wightman correlators (2.38) are now written in terms of fields rather than position operators. Namely, one has

$$\begin{aligned} D^>(x) &= \langle\varphi(x)\varphi(0)\rangle_\beta\,, \\ D^<(x) &= \langle\varphi(0)\varphi(x)\rangle_\beta\,. \end{aligned} \tag{3.7}$$

The Wightman correlators in (3.7) can also be defined for **local operators** of the field theory. Local operators are defined as normal-ordered products of the fields at a fixed space-time point. The normal ordering prescription (which we will denote as $: \cdots :$) puts all creation operators to the right and all destruction operators to the left. Through this process one can get rid of the problematic contact terms ($\sim \delta(0)$ vacuum divergent terms) and obtain well-defined operators. For instance, in a scalar field theory, we may define local operators as

$$O_n(x) =: \varphi^n(x) : \tag{3.8}$$

and hence we could define the correlation function of any local operator in the style of (3.7). For instance, the (<) Wightman two-point function for the $n = 2$ local operator is

$$D_2^<(x) = \langle : \varphi^2(0) :: \varphi^2(x) : \rangle_\beta \,. \tag{3.9}$$

Looking at (3.9) one can very easily convince oneself that it is closely related to a Wightman self-energy. Beyond intuition, by expanding in creation/annihilation operators one would get to $D_2^<(x) \sim (D^<(x))^2$ (symmetry factor aside), which is the very definition of a self-energy bubble. Thus we find an important message: self energies may be understood as correlators of quadratic local operators in a cubic theory (we will see). Namely, for a thermal scalar theory and aside from symmetry factors that may appear,

$$\Pi^<(x) \sim \langle : \varphi^2(0) :: \varphi^2(x) : \rangle_\beta \,. \tag{3.10}$$

It is also worth stressing that every property of the Wightman correlators we have been discussing also holds for the Wightman two-point functions of local operators.

This is of course an sketchy treatment of one of the many delicate corners of QFTs, and the reader should know that this issue extends to much deeper grounds. However, for our interests, this depth level will be enough.

3.3 Self-energies and emission rates for various theories

In this section we will combine what we know about self-energies and field theories at finite temperature to derive a general expression to compute emission rates. We begin by illustrating a scalar theory, and then we will generalise it to quantum electrodynamics (QED for short) in order to make full sense of the case of coupling to gravity we are ultimately interested in.

The common playground is a system of strongly coupled thermal particles in a bath interacting weakly with another kind of massive or massless particles. We will describe the production of a weakly interacting particle in a general case for each theory.

3.3.1 Scalar theory

In this toy scenario, the thermal particles in the bath are embedded in the field $\varphi(x)$ and the weakly interacting particles are embedded in $\Phi(x)$. We assume a cubic weak interaction term $\mathscr{L}_I \sim \lambda\Phi(x)\varphi^2(x)$ and a quartic strong coupling term $g\varphi^4(x)$. The transition we are interested in is $(i,0) \to (f,q)$, where i, f refer to the thermal states and $q_0 > 0$ is the energy of the produced Φ-particle. Let us assume no Φ-particle in the initial state for simplicity.

The interaction matrix in old-fashioned QFT is $S = T\left[e^{i\int d^4x \mathscr{L}_I(x)}\right]$, which for our interaction Lagrangian yields $S = 1 + i\lambda\int d^4x T\left[\Phi(x)\varphi^2(x)\right]$ up to first order in the weak coupling constant. If we sandwich the non-trivial term with the initial and final states, we can write

$$S_{if}(q) = i\lambda \int d^4x \left\langle f, q\right| : \Phi(x)\varphi^2(x) : \left|i, 0\right\rangle . \tag{3.11}$$

Upon writing $\langle f, q| = \langle f, 0| \hat{a}_\Phi(\mathbf{q})$ and contracting the annihilation operator with the $\Phi(x)$ operator inside of the $: \cdots :$ in (3.11) yields the transition amplitude

$$S_{if}(q) = i\lambda \int d^4x e^{iqx} \left\langle f\right| : \hat{\varphi}^2(x) : \left|i\right\rangle , \tag{3.12}$$

where we have recovered the hat for the operators and discarded the zeros in the brakets. If we now square the modulus of (3.12) and use translational invariance, we end up with

$$\begin{aligned} |S_{fi}(q)|^2 &= \lambda^2 \int d^4x \int d^4y e^{iq(x-y)} \left\langle f\right| : \hat{\varphi}^2(x-y) : \left|i\right\rangle \left\langle i\right| : \hat{\varphi}^2(0) : \left|f\right\rangle \\ &= \Omega\lambda^2 \int d^4x e^{iqx} \left\langle f\right| : \hat{\varphi}^2(x) : \left|i\right\rangle \left\langle i\right| : \hat{\varphi}^2(0) : \left|f\right\rangle , \end{aligned} \tag{3.13}$$

which is the probability of the transition. In the last step of (3.13) we have introduced the space-time volume in which the interaction takes place, Ω. Now we have to take into account the statistical weight that initial states possess in a canonical ensemble. Using the fact that $e^{-\beta\hat{H}}|i\rangle = e^{-\beta E_i}|i\rangle$ and the cyclicity of the trace, we can write

$$\frac{1}{\Omega}\frac{1}{Z(\beta)}\sum_{i,f} e^{-\beta E_i}|S_{fi}(q)|^2 = \lambda^2 \int d^4x e^{iqx} \langle : \hat{\varphi}^2(0) :: \hat{\varphi}^2(x) : \rangle_\beta . \tag{3.14}$$

The Fourier transform in the r.h.s. of (3.14) is indeed a Wightman self-energy, since it loosely amounts to $\sim (D^<(x))^2$. Integrating for all possible momenta of the outgoing particle with the Lorentz invariant measure yields a decay rate, which is proportional to a Wightman self-energy, namely:

$$\frac{1}{\Omega}\frac{1}{Z(\beta)}\sum_{i,f} e^{-\beta E_i}|S_{fi}(q)|^2 = \Pi^<(q). \tag{3.15}$$

Hence, we can finally write the total production rate of weakly interacting particles per unit time and volume for this scalar model:

$$\Gamma_\Phi = \int \frac{d^3q}{2q_0(2\pi^3)} \Pi^<(q), \tag{3.16}$$

as well as its differential form with respect to the momentum q

$$q_0 \frac{d\Gamma_\Phi}{d^3q} = \frac{1}{2} \frac{\Pi^<(q)}{(2\pi)^3}. \tag{3.17}$$

Note that the self-energy is nothing but the correlation function of the local operator $\varphi^2(x)$, and that this is a straightforward generalisation of the Wightman two-point functions we wrote for $O_1(x)$ local operators in (2.38), but now with quadratic local operators. This implies that the relation (2.90) also holds for the self-energy correlator, and hence we can write

$$q_0 \frac{d\Gamma_\Phi}{d^3q} = \frac{1}{2} \frac{\Pi^<(q)}{(2\pi)^3} = -\frac{f(k_0)\,\mathrm{Im}\{\Pi_R(q)\}}{(2\pi)^3}. \tag{3.18}$$

Note that the first expression on the r.h.s. of (3.18) is indeed the energy density emission rate per unit time. To see this, consider the number of emitted particles per unit time and volume $\Gamma_\Phi \sim \frac{N_\Phi}{V\Delta t}$ with N_Φ the number of emitted Φ-particles and Δt the period of time in which the interaction takes place such that $\Omega = V\Delta t$ (with V the space volume). When we consider the quantity $\frac{q_0 N_\Phi}{V} = \frac{E_\Phi}{V} \equiv \rho_\Phi$ we are indeed considering the energy density emission in the form of Φ-particles per unit time, that is, ρ_Φ. Hence, from (3.16), we can write

$$\frac{d\rho_\Phi}{dtd^3q} = \frac{1}{2} \frac{\Pi^<(q)}{(2\pi)^3} = -\frac{f(k_0)\,\mathrm{Im}\{\Pi_R(q)\}}{(2\pi)^3}. \tag{3.19}$$

3.3.2 Quantum electrodynamics

The QED scenario is slightly more involved because it is a more realistic one. We suppose a thermal background of strongly coupled quarks and gluons forming a plasma which interact weakly with the photons and leptons that obey the rules of QED. The transition we are interested in is $(i, 0) \to (f, \gamma)$, where γ describes a photon with four-momentum Q.

We attack this problem with the same strategy we used for the scalar case. In this case the interaction Lagrangian for QED is $\mathscr{L} \sim -ej^\mu A_\mu$ (with j^μ the $U(1)$ conserved current and e the QED coupling constant). We can write the transition amplitude for a certain spin state labeled by (λ) as

$$S_{if}^{(\lambda)}(Q) = -ie \int d^4x \, \langle f, Q| :A_\mu^{(\lambda)}(x) j^\mu(x): |i, 0\rangle \,. \tag{3.20}$$

The only difference with the above section is the presence of the polarization vector $\epsilon_\mu^{(\lambda)}(Q)$ in the ladder operator expression of the photon field, which accounts for the spin state $\{-1, 1\}$ of the massless photon. That is, we now use $\langle f, Q| = \langle f, 0| \hat{a}_\gamma(\mathbf{Q})\epsilon_\mu^{(\lambda)*}(Q)$. Having said this, one just repeats the scalar case computation but now also summing over spin states and gets

$$\frac{1}{\Omega}\frac{1}{Z(\beta)}\sum_{i,f,\lambda} e^{-\beta E_i}|S_{fi}(Q)|^2 = -e^2\int d^4x e^{iQx}\left[\eta_{\mu\nu}\left\langle :j^\mu(0)::j^\nu(x):\right\rangle_\beta\right], \tag{3.21}$$

where we have used that

$$\sum_\lambda \epsilon_\mu^{(\lambda)}(Q)\epsilon_\nu^{(\lambda)*}(Q) = \eta_{\mu\nu}. \tag{3.22}$$

The manifest similarity between (3.21) and (3.14) invites us to deduce that (3.21) is indeed an expression for a Wightman self-energy. Since the QED current reads $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ (for ψ, $\bar{\psi}$ the fermions of the theory), the diagramatic interpretation is a fermion/anti-fermion bubble closing in space-time points $0, x$ with ($<$)-propagators. Hence, an expression for the photon emission rate can be obtained upon a suitable substitution of the new QED self-energy

$$\Pi^<_{QED}(Q) = \eta_{\mu\nu}\Pi^{<\mu\nu}(Q) = e^2\int d^4x e^{iQx}\eta_{\mu\nu}\left\langle :j^\mu(0)::j^\nu(x):\right\rangle_\beta \tag{3.23}$$

in the expressions (3.16) and (3.17), finally yielding the photon emission rate. Hence, the photon differential emission rate reads

$$q_0\frac{d\Gamma_\gamma}{d^3q} = -\frac{1}{2}\frac{\Pi^<_{QED}(Q)}{(2\pi)^3} = \frac{f(k_0)\eta^{\mu\nu}\,\mathrm{Im}\{\Pi^R_{\mu\nu}(Q)\}}{(2\pi)^3}, \tag{3.24}$$

where there is a minus sign difference with (3.18) because of the consideration of $-e^2$ as coupling constant. The expression for the photon energy density is obtained through the same arguments that we obtained (3.19), and hence we can write

$$\frac{d\rho_\Phi}{dtd^3q} = -\frac{1}{2}\frac{\Pi^<_{QED}(Q)}{(2\pi)^3} = \frac{f(k_0)\eta^{\mu\nu}\,\mathrm{Im}\{\Pi^R_{\mu\nu}(Q)\}}{(2\pi)^3}, \tag{3.25}$$

3.4 A lesson for the future

We have compiled comprehensive models of thermal emission of spin-0 and spin-1 weakly interacting massless particles, and there is a manifest similarity between the expressions of their emission rates (3.19) and (3.25).

In the scalar case, we have seen how the Fourier transform of the φ^2 correlator (see (3.14), a.k.a. Φ-particle self-energy) gives the emission rate of Φ-particles from a cubic interaction term of the form $\sim \varphi^2\Phi$. In the QED case we have seen how the Fourier transform of the current (j_μ) correlator (see (3.21), a.k.a. photon self-energy) gives the emission rate of photons from an interaction term of the form $\sim j_\mu A^\mu$, with $j^\mu = \bar{\psi}\gamma^\mu\psi$ a fermionic bilinear, which is again a cubic vertex of two fermions and a photon.

These terms may also be interpreted like a source $J(x) = \Phi(x)$, $J_\mu(x) = A_\mu(x)$ coupled to a local operator of the QFT $O(x) =: \varphi^2(x) :$, $O^\mu(x) =: j^\mu(x) :$. They can also be derived by functional differentiation of a generating functional with respect to the source. Take for simplicity the scalar case. Upon defining the following functional

$$Z_\Pi[J] = \langle e^{i\int d^4x J(x)O(x)}\rangle_\beta\,, \tag{3.26}$$

one simply takes two functional derivatives with respect to the source and gets the desired output: the two point function of the local operator $O(x)$. Namely, one does

$$(-i)^2 \frac{\delta Z_\Pi[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \langle O(x_1)O(x_2)\rangle_\beta = \Pi^<(x_2 - x_1) = \Pi^>(-x_2 + x_1), \tag{3.27}$$

where we used translation invariance. Note how we flipped the usual source-operator character of each factor in order to define intuitive systematics.

The case we are interested in is the spin-2 case. We want to compute the emission rate of gravitational waves (GWs), or conversely, of gravitons, which is just the name that one gives to a metric perturbation propagating in a background space. In this case, we can intuitively deduce that the interactive term will amount $\sim T^{\mu\nu}h_{\mu\nu}$, with $T^{\mu\nu}$ the local operator and $h_{\mu\nu}$ the source. In the following chapter we will deduce in detail the form of the correlator that gives the formal expression of the emission rate of GWs. The graviton case deserves a chapter on its own, since the formalism we will develop to describe it has two uses: it describes the physical GWs to whom our emission rate makes reference and it also describes the metric perturbations in an AdS background.

3.5 The complex scalar field

As a final exercise, let us study a complex scalar field at finite temperature. Complex scalar fields are invariant under global $U(1)$ symmetry transformations. The Minkowskian action describing a massless complex scalar field is

$$S = \int d^4x \partial^\mu \phi^\dagger(x) \partial_\mu \phi(x), \tag{3.28}$$

which is invariant under a transformation

$$\phi(x) \to \phi'(x) = e^{-i\alpha} \phi(x) \tag{3.29}$$

if α is a real constant parameter. Indeed, through Noether theorem one finds that there is a conserved current

$$j^\mu(x) = i \left[\phi^\dagger(x) \partial^\mu \phi(x) - \phi(x) \partial^\mu \phi^\dagger(x) \right] \tag{3.30}$$

and a conserved charge (which is *grosso modo* equivalent to the number of particles minus antiparticles)

$$N = \int d^3x j^0(x) = \int d^3x \left[\pi^\dagger(x) \phi^\dagger(x) - \pi(x) \phi(x) \right], \tag{3.31}$$

where $\pi^{(\dagger)}(x) = \partial^0 \phi^{(\dagger)}(x)$ are the canonical momenta.

If we allow for a local dependence $\alpha(x)$, the action (3.28) is no longer invariant under the transformation

$$\phi \to \phi' = e^{-i\alpha(x)} \phi. \tag{3.32}$$

Indeed, if we consider a local infinitesimal transformation $\phi' \sim \phi - i\alpha(x)\phi$, the variation of the action under this transformation is (up to linear order in α)

$$\delta S = j^\mu(x) \partial_\mu \alpha(x) \equiv j^\mu(x) A_\mu(x). \tag{3.33}$$

Hence, if we want a theory that is invariant under local $U(1)$ transformations we require an action

$$S = \int d^4x \left[\partial^\mu \phi^\dagger(x) \partial_\mu \phi(x) - j^\mu(x) A_\mu(x) \right]. \tag{3.34}$$

In other words, the global $U(1)$ current couples to a gauge field. There is another convenient way to write the action through a covariant derivative which reads

$$S = \int d^4x \left[(D^\mu \phi(x))^\dagger (D_\mu \phi(x)) \right], \tag{3.35}$$

where we have defined the covariant derivative $D_\mu = \partial_\mu - iA_\mu(x)$. If one performs the Euclidean continuation ($t \to -i\tau$) of (3.35) with a gauge field defined by its only non-vanishing component $A_t \neq 0$, it can be written as

$$\begin{aligned} iS_E(\beta, A_t) = i \int_0^\beta \int d^3x d\tau' \left\{ [\partial_\tau + A_t] \phi^\dagger(-i\tau, \mathbf{x}) [\partial_\tau - A_t] \phi(-i\tau', \mathbf{x}) \right\} \\ + i \int_0^\beta \int d^3x d\tau' \left\{ \partial_i \phi^\dagger(-i\tau', \mathbf{x}) \partial_i \phi(-i\tau', \mathbf{x}) \right\}. \end{aligned} \tag{3.36}$$

Since (3.34) is completely equivalent to (3.35), one may also write (for the A_t gauge field)

$$\begin{aligned} iS_E(\beta,\mu) = i\int_0^\beta \int d\tau d^3x \left[\partial_t\phi^\dagger(-i\tau,\mathbf{x})\partial_t\phi(-i\tau,\mathbf{x}) + \partial_t\phi(-i\tau,\mathbf{x})\partial_t\phi^\dagger(-i\tau,\mathbf{x})\right] \\ + i\int_0^\beta d\tau A_t N(-i\tau,\mathbf{x}) \equiv iS_E^0(\beta,\mu) + i\int_0^\beta d\tau N(\tau,\mathbf{x})A_t \end{aligned} \tag{3.37}$$

From our discussion on the previous chapter, it is clear that the Euclidean continuation of this new action will have an interpretation in terms of QSM. Indeed, let us consider a quantum system in the grand-canonical ensemble where the particle number is allowed to fluctuate. From QSM we know that the partition function for such system is written as

$$Z(\beta,\mu) = \mathrm{Tr}\Big\{e^{-\beta(\hat{H}-\mu\hat{N})}\Big\}. \tag{3.38}$$

The chemical potential appears to tell us that the system is now opened, and that is why it couples to the number operator. In other words, now the energy E_i of a given state $|\psi_i\rangle$ is conditioned by the number of particles that may have exit the system (or entered it). The expression of the partition function as a path integral in periodic Euclidean time is not as easy to obtain as it was in the canonical ensemble because $\hat{N}$ (the number operator that comes from quantizing and normal-order (3.31)) depends on the time derivative of the field, and hence one has use the Hamiltonian formalism to compute the path integral (it is done in Le Bellac chapter 3 [13]). If one does this, one proves that the partition function (3.38) is

$$\begin{aligned} Z(\beta,\mu) = \int \mathscr{D}\phi \exp\Big\{-\int_0^\beta \int d^3x d\tau' \left\{\left[\partial_\tau + \mu\right]\phi^\dagger(-i\tau,\mathbf{x})\left[\partial_\tau - \mu\right]\phi(-i\tau',\mathbf{x})\right\}\Big\} \\ \times \exp\Big\{-\int_0^\beta \int d^3x d\tau' \left\{\partial_i\phi^\dagger(-i\tau',\mathbf{x})\partial_i\phi(-i\tau',\mathbf{x})\right\}\Big\}, \end{aligned} \tag{3.39}$$

which is completely equivalent to the path integral that would emerge from $e^{-S_E(\beta,\mu)}$ by exponentiating (3.36) and substituting $A_t = \mu$. That the gauge field only has a temporal component is to be expected because the chemical potential is a quantity that makes sense in QSM where there are no spatial components. To sum up, we may write

$$Z(\beta,\mu) = \int \mathscr{D}\phi \exp\{-S_E(\beta,\mu)\} = \int \mathscr{D}\phi \exp\{-S_E(\beta)\} \exp\Big\{-\int_0^\beta d\tau N(\tau,\mathbf{x})\mu\Big\}, \tag{3.40}$$

where we used both (3.36) and (3.37). It is clear that we can understand μ as a source for the particle number.

The message is thus quite clear: if the action has a local gauge symmetry with gauge field $A_t \neq 0$, its Euclidean continuation amounts to a system at finite temperature T and chemical potential $A_t = \mu$. This kind of system is usually referred to as a finite temperature and density field theory. It is said to be at finite density because the chemical potential quantifies the number of particles that enter the system. If it is large $\mu \gg T$, the system is said to be at high density.

Chapter 4

Gravitational waves

In this chapter we will properly define GWs as fluctuations to the metric and we will get a flavour of their quasi-particle interpretation as gravitons. GWs are ondulations in space-time that propagate at the speed of light. They are generated in strong gravity regimes, but one can consider their propagation far from sources in the weak gravity regime in which the GWs simply become small corrections to the flat Minkowski metric. If one develops general relativity (GR) up to linear order in the perturbations, the output is known as linearised gravity and it helps to interpret a GW as a particle.

We will add a thermal source to the linearised equations of motion (EOMs from now on) and express the emission of GWs from the thermal source in terms of a correlator of the stress-energy tensor representing the thermalized matter source. The correlator will emerge naturally since we will have to thermally average every possible configuration of the source.

4.1 Linearised gravity

Let us consider a perturbation to a 4-dimensional Minkowski metric, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

with $|h_{\mu\nu}| \ll 1$. The indices of the perturbations $h_{\mu\nu}$ are lowered and raised with the η and the inverse metric picks a minus on the h term.

With the metric (4.1) one can go all the way through the Christoffel symbols, the Riemann tensor, the Ricci tensor and the Ricci scalar to get to the Einstein tensor up to linear order in $h_{\mu\nu}$ and write the Einstein equations in linearised gravity. Recall that the Einstein equations read

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}\,, \tag{4.2}$$

with $R_{\mu\nu}$ and R the Ricci tensor and scalar and $T_{\mu\nu}$ the stress energy tensor representing the matter content. Recall that $\kappa^2 = 8\pi G_N$ with G_N the newtonian gravitational constant. Our Einstein equations are

$$G_{\mu\nu} = -\frac{1}{2}\left(\Box \bar{h}_{\mu\nu} - \partial_\rho \partial_\nu \bar{h}_\mu{}^\rho - \partial_\mu \partial_\rho \bar{h}^\rho{}_\nu + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}\right) = \kappa^2 T_{\mu\nu}\,, \tag{4.3}$$

where the redefinition $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\rho{}_\rho$ has been used. At first, this expression strikes as a bit messy. We can use the invariance under diffeomorphisms of the Einstein tensor to render (4.3) more aesthetic. Indeed, it is easy to show that under an infinitesimal diffeomorphism $\tilde{x}^\mu = x^\mu + \xi^\mu(x)$ a metric tensor transforms as

$$g_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) \frac{\partial x^\rho}{\partial x^\mu} \frac{\partial x^\sigma}{\partial x^\nu} = g_{\mu\nu}(x) - g_{\rho\nu}(x) \partial_\mu \xi^\rho(x) - g_{\mu\rho}(x) \partial_\nu \xi^\rho(x) + \mathscr{O}(\xi^2). \tag{4.4}$$

It is also straightforward to prove that the linearized version of $G_{\mu\nu}$ is invariant under such transformations. Now we can choose a particular expression for $\xi^\mu(x)$ that simplifies (4.3). We are fixing the gauge freedom. Indeed, by looking at (4.3) one can see that it would be perfect to cancel the $\partial\bar{h}$-terms from the Einstein tensor. More precisely, we would require $\partial_\mu \bar{h}^{\mu\nu}(\tilde{x}) = 0$ for the transformed perturbations. Such simplification can be achieved by choosing the so-called Lorenz gauge condition. The Lorenz gauge condition is

$$\Box \xi^\mu(x) = \partial_\mu \bar{h}^{\mu\nu}(x). \tag{4.5}$$

With this condition, it is not hard to show that[1]

$$\partial_\mu \bar{h}^{\mu\nu}(\tilde{x}) = \Box \xi^\nu(x) - \Box \xi^\nu(x) = 0 \tag{4.6}$$

is fulfilled for all $\xi^\nu(x)$ and hence there is a residual degree of gauge freedom to be fixed by the diffeomorphism function. We choose

$$\Box \xi^\nu(x) = 0. \tag{4.7}$$

[1] The metric perturbation transforms as $\bar{h}_{\mu\nu}(\tilde{x}) = h_{\mu\nu}(x) - \Box \xi_\nu(x) - \partial_\nu \partial_\rho \xi^\rho(x)$, and hence using (4.5) we find (4.6).

With all these specifications, we can rewrite the Einstein equations (4.3) in a much simpler way. Indeed, the resulting equation is a wave equation for the perturbations

$$\Box\bar{h}_{\mu\nu}(\tilde{x}) = -2\kappa^2 T_{\mu\nu}(\tilde{x}), \tag{4.8}$$

where we have written the tilde in $\tilde{x}$ to emphasise that the variables have been gauge transformed.

4.1.1 The TT gauge

Let us for the moment work in the vacuum, $T_{\mu\nu} = 0$. If that is the case, a plane wave solution for the perturbation in (4.8) may be found, yielding

$$\bar{h}_{\mu\nu}(\tilde{x}) = \tilde{A}_{\mu\nu}e^{ik_\rho x^\rho}, \tag{4.9}$$

which implies that GWs propagate at the speed of light in vacuum due to the dispersion relation $k^\rho k_\rho = 0$ that stems from the computation. We can also solve for of the diffeomorphism vector field such that $\Box\xi^\mu(x) = 0$. We also find a solution with $k^\rho k_\rho = 0$ that reads

$$\xi^\mu(x) = B^\mu e^{ik_\rho x^\rho}. \tag{4.10}$$

Now we can use (4.4) to determine the form of $\tilde{A}_{\mu\nu}$ in terms of the old $A_{\mu\nu}$ from (4.9), yielding

$$\tilde{A}_{\mu\nu} = A_{\mu\nu} - i\eta_{\nu\rho}k_\mu B^\rho - i\eta_{\mu\rho}k_\nu B^\rho + i\eta_{\mu\nu}k_\rho B^\rho. \tag{4.11}$$

Now we can use the freedom to choose the form of the components of B^μ. We impose that B^μ is such that

$$\begin{aligned} \tilde{A}^\mu{}_\mu &= A^\mu{}_\mu + 2ik_\mu B^\mu = 0, \\ u^\nu\tilde{A}_{\mu\nu} &= u^\nu A_{\mu\nu} - iu_\rho k_\mu B^\rho - iu^\nu\eta_{\mu\rho}k_\nu B^\rho + iu_\mu k_\rho B^\rho = 0. \end{aligned} \tag{4.12}$$

In other words, we choose B^μ such thath $\tilde{A}_{\mu\nu}$ is traceless and transverse to a unit vector we define as u^μ which describes the direction of the four-velocity of an observer. There is yet another constraint to the components of the tensor amplitude of the perturbation that stems from (4.6),

$$k^\nu\tilde{A}_{\mu\nu} = 0. \tag{4.13}$$

Technically, (4.12) is a system of 5 equations for 4 unknowns B^μ. However, from (4.13) we can show that a linear combination of the 4 equations in the second line of (4.12) vanishes,

$k^\mu u^\nu \tilde{A}_{\mu\nu} = 0$, and hence only three of those equations are independent. Therefore there exists a unique solution for B^μ. Now that we know that the system (4.12) is well defined and has a unique solution, we can affirm that there is a certain B^μ such that

$$\begin{aligned} k^\mu \tilde{A}_{\mu\nu} &= 0, \\ u^\mu \tilde{A}_{\mu\nu} &= 0, \\ \tilde{A}^\mu{}_\mu &= 0. \end{aligned} \tag{4.14}$$

The conditions (4.14) define the so called **transverse traceless gauge** (TT gauge for short). The tracelessness condition allows us to get rid of the bar in the notation, $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$. We introduce the notation TT to reference the TT gauge. Now that there is no confusion possible, we can at last suppress the tildes in the notation.

We may now choose an observer in the rest frame, $u^\mu = (1, \vec{0})$ and assume that the wave propagates along the z-axis, yielding $k^\mu = (\omega, 0, 0, \omega)$. With these considerations, one easily sees from (4.14) that the only non-vanishing components of the perturbations are those with $A^{TT}_{xy} = A^{TT}_{yx}$ and $A^{TT}_{xx} = -A^{TT}_{yy}$. Therefore, in the TT gauge, Einstein equations in vacuum read

$$\Box h^{TT}_{ij}(t, z) = 0, \tag{4.15}$$

and they admit a simple plane wave solution

$$h^{TT}_{ij}(t, z) = A^{TT}_{ij} e^{-i\omega t + i\omega z} \tag{4.16}$$

where the latin indices refer to the non-vanishing spatial components.

4.2 Adding a source

Now we simply want to add an energy-momentum tensor as a source for (4.15), also in the TT gauge. Namely

$$\Box h^{TT}_{ij}(t, \mathbf{x}) = -2\kappa^2 T^{TT}_{ij}(t, \mathbf{x}), \tag{4.17}$$

where we take the transverse and traceless components of the energy-momentum tensor. Latin indices denote spatial components, since the $h^{TT}_{0\mu} = 0$. Note that the solution to (4.17) will not be as simple as the plane wave in (4.16). We define the Fourier transform

$$h^{TT}_{ij}(t, \mathbf{x}) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} h^{TT}_{ij}(\omega, \mathbf{k}), \tag{4.18}$$

which allows writing the following equation in Fourier space (we also Fourier-transform the energy-momentum tensor)

$$h_{ij}^{TT}(\omega,\mathbf{k}) = 2\kappa^2 \frac{T_{ij}^{TT}(\omega,\mathbf{k})}{(\omega+i\epsilon)^2-\mathbf{k}^2}. \tag{4.19}$$

In (4.19) we have picked the physical retarded solution, with $\epsilon \to 0$. Also recall that $k^\mu k_\mu = 0$.

Note that we have tiptoed the fact that we do not really know how to write $T_{\mu\nu}$ in the TT gauge. This can be done by defining a projector that selects the TT components of a tensor. In order not to bomb the reader with a page of algebra, we prooflessly give the reader the projector. The projector reads

$$\Lambda_{\mu\nu\rho\sigma} \equiv P_{\mu\rho}P_{\nu\sigma} - \frac{1}{2}P_{\mu\nu}P_{\rho\sigma}\,, \tag{4.20}$$

with $P_{\mu\nu}$ the transverse projector

$$\begin{aligned} P_{ij} &= \eta_{ij} - \frac{k_i k_j}{k^2}, \\ P_{0\mu} &= \eta_{0\mu} - u_0 u_\mu = 0, \forall \mu. \end{aligned} \tag{4.21}$$

From here, it is easy to see that the transverse projector gets rid of all time and z-components of a tensor, while their combination (4.20) renders it traceless[2]. Now we can write the spatial components of the TT gauge energy-momentum tensor as

$$T_{ij}^{TT}(\omega,\mathbf{k}) = \Lambda_{ijmn}T_{mn}(\omega,\mathbf{k}). \tag{4.22}$$

4.2.1 The classical energy of a gravitational wave

Before going forward with the computation of the emission rate for GWs, let us make some brief comments on how can one obtain an expression for the energy carried away from a system by a GW, which is obviously crucial to our computation.

Up to this point we have been working up to linear order on the metric perturbations. It turns out that in order to obtain the energy of a GW one has to consider second order terms. Even though the computations become rather messy in doing so, the concept behind this

[2]It is easy to check that for any tensor $V^{\rho\sigma}$ one gets $\eta^{\mu\rho}\eta^{\mu\sigma}\Lambda_{\mu\nu\rho\sigma}V^{\mu\nu} = 0$.

extension is in fact pretty simple: the energy of a GW can be deduced from the curvature that the GW itself generates on spacetime. This is indeed the very definition of second order calculations on linearised gravity. More concretely, up to now our Einstein equations have been

$$G^{(1)}_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{4.23}$$

but they could have been extended up to any order in $h_{\mu\nu}$. Up to second order, one has

$$G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{4.24}$$

where $G^{(2)}_{\mu\nu}$ can be read off as a correction to the stress-energy tensor, yielding

$$G^{(1)}_{\mu\nu} = \kappa^2 \left[T_{\mu\nu} + \mathscr{T}_{\mu\nu}\right], \tag{4.25}$$

where $\mathscr{T}_{\mu\nu} = -\frac{G^{(2)}}{\kappa^2}$ is the so-called Isaacson stress-energy pseudo-tensor, and believe me when I tell you it is a true pain to compute. It is called a pseudo-tensor because, for starters, it is not even gauge invariant. However, there is a clever trick to obtain a gauge invariant quantity out of this object. Although smart, the procedure is nothing but an average (sending fluctuations to 0), namely

$$G^{(2)} = \left[G^{(2)} - \left\langle G^{(2)} \right\rangle\right] + \langle G^{(2)} \rangle \to \left\langle G^{(2)} \right\rangle \tag{4.26}$$

Since this computation would take us far away from the main goals of this work, the reader is referred to [14] for details. The final result is a gauge invariant quantity that can of course be written in the TT gauge. One can deduce that, in the TT gauge, the averaged gauge-invariant Isaacson energy-momentum pseudo-tensor can be written as

$$\mathscr{T}^{TT}_{\mu\nu} = \eta^{\rho\lambda}\eta^{\sigma\tau}\frac{1}{4\kappa^2}\left\langle \partial_\mu h^{TT}_{\lambda\tau}\partial_\nu h^{TT}_{\rho\sigma}\right\rangle, \tag{4.27}$$

where $\langle \ldots \rangle$ denotes a time average over an observation period that is long compared to the frequency of the wave (or conversely, over a volume that is big compared to the wavelength of the wave). It is straightforward to read off the averaged energy density carried out by a GW from (4.27), since it is the 00-th component of its stress-energy tensor. We denote by $\frac{d\bar{E}_{GW}}{d^3x} \equiv \mathscr{T}^{TT}_{00}$ the energy density carried by the GW, and we write its expression in terms of the spatial components of the metric perturbation:

$$\frac{d\bar{E}_{GW}}{d^3x} = \frac{1}{4\kappa^2}\left\langle \dot{h}^{TT}_{ij}(t,\mathbf{x})\dot{h}^{TT}_{ij}(t,\mathbf{x})\right\rangle. \tag{4.28}$$

Needless to say that (4.28) is indeed gauge invariant.

We can Fourier transform an integrate (4.28) to obtain

$$\bar{E}_{GW} = \frac{1}{4\kappa^2}\int \frac{d^3k}{(2\pi)^3}\left\langle \dot{h}_{ij}^{TT}(t,-\mathbf{k})\dot{h}_{ij}^{TT}(t,\mathbf{k})\right\rangle, \tag{4.29}$$

where the minus sign on $\dot{h}_{ij}^{TT}(t,-\mathbf{k})$ stems from a delta integral. With (4.29) in our hands we are ready to go forward.

4.3 Thermal emission of gravitational waves

In this section we work our way to obtaining the emission rate of GWs from a thermal source. In order to be accurate, we consider two connecting ways of getting there. In the first subsection, the emission rate of GWs is obtained from a purely classical approach to GWs. In the second subsection the concept of graviton arises as we treat GWs as particles in a QFT with κ as coupling constant.

4.3.1 Classical treatment

From (4.29) we read an expression for the classical energy that flows out of a system in the form of GWs. We may take our solution for the perturbation that represents the GWs in terms of the stress-energy tensor (4.19) and use it to compute the energy with (4.29). We can write

$$h_{ij}^{TT}(t,\mathbf{k}) = 2\kappa^2\int \frac{d\omega}{2\pi}e^{-i\omega t}\frac{T_{ij}^{TT}(\omega,\mathbf{k})}{(\omega+i\epsilon)^2-\mathbf{k}^2}. \tag{4.30}$$

Now we just have to plug in the Fourier transform $T_{ij}^{TT}(\omega,\mathbf{k}) = \int dt' e^{i\omega t'}T_{ij}^{TT}(t',\mathbf{k})$ and solve the integral through the residues in the retarded poles. In doing so, one obtains

$$h_{ij}^{TT}(t,\mathbf{k}) = 2\kappa^2\int_{-\infty}^{t} dt' \frac{\sin\left(\omega(t-t')\right)}{\omega}T_{ij}^{TT}(t',\mathbf{k}). \tag{4.31}$$

Now we simply plug (4.31) in (4.29) and find

$$\begin{aligned}\bar{E}_{GW} &= \kappa^2\left\langle\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt''\cos\left[\omega(t-t')\right]\cos\left[\omega(t-t'')\right]T_{ij}^{TT}(t',-\mathbf{k})T_{ij}^{TT}(t'',\mathbf{k})\right\rangle\\ &= \frac{\kappa^2}{2}\int\frac{d^3k}{(2\pi)^3}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt''\Bigg[\left\langle\cos\left(\omega(2t-t'-t'')\right)\right\rangle + \left\langle\cos\left(\omega(t'-t'')\right)\right\rangle\Bigg]\\ &\times T_{ij}^{TT}(t',-\mathbf{k})T_{ij}^{TT}(t'',\mathbf{k}),\end{aligned} \tag{4.32}$$

where the average $\langle \ldots \rangle$ is taken over the a long period in the variable t. The average helps us get rid of the fast oscillating terms (fluctuations) encoded in the first term of (4.32). Since the second term is functionally independent of t, the average throws the oscillating term away so that we can finally write

$$\bar{E}_{GW} = \frac{\kappa^2}{2} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \cos\left[\omega(t'-t'')\right] T_{ij}^{TT}(t', -\mathbf{k}) T_{ij}^{TT}(t'', \mathbf{k}). \tag{4.33}$$

Up to now, no assumptions on the source have been made. The expression (4.33) is valid for any source, whether it is thermalized or not. In our case, we consider a source formed by thermally bound particles that interact weakly with the produced GWs but strongly among them, just like we did in the scalar and QED cases. Note that (4.33) gives the energy for only one particular configuration of the thermal ensemble, and hence we want to thermally average the configurations of the ensemble in order to obtain the full picture of the GW emission. Also, note that the spectrum of the GWs will not be thermal, but rather it will contain information about the thermal source that generates them. Hence, we will express the emission rate in terms of a thermal correlator. Namely, we define the following quantity

$$\mathscr{C}(t'-t'', \mathbf{k}) \equiv \left\langle T_{ij}^{TT}(t', -\mathbf{k}) T_{ij}^{TT}(t'', \mathbf{k}) \right\rangle_\beta . \tag{4.34}$$

Assuming traslational invariance, one may write (4.34) by Fourier-transforming in the spatial coordinates as

$$\begin{aligned} \mathscr{C}(t'-t'', \mathbf{k}) &= \int d^3x \int d^3y \left\langle T_{ij}^{TT}(t'-t'', \mathbf{x}-\mathbf{y}) T_{ij}^{TT}(0,0) \right\rangle_\beta e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \\ &= V \int d^3x \left\langle T_{ij}^{TT}(t'-t'', \mathbf{x}) T_{ij}^{TT}(0,0) \right\rangle_\beta e^{-i\mathbf{k}\mathbf{x}}, \end{aligned} \tag{4.35}$$

where V is the purely spatial volume of the thermal system. Now we can take the thermal average of (4.33) to write

$$\left\langle \bar{E}_{GW} \right\rangle_\beta = \frac{\kappa^2}{4} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dt'' \left[e^{i\omega(t'-t'')} + e^{-i\omega(t'-t'')} \right] \mathscr{C}(t'-t'', \mathbf{k}). \tag{4.36}$$

After taking a time-derivative and a derivative with respect to $\mathbf{k}$ in (4.36) one finds that, since $\frac{d}{dx}\left(\int_a^x F(y)dy\right) = F(x)$,

$$\begin{aligned} \frac{d\left\langle \bar{E}_{GW} \right\rangle_\beta}{dt d^3k} &= \frac{1}{2} \frac{\kappa^2}{(2\pi)^3} \int_{-\infty}^{+\infty} d\tau e^{i\omega\tau} \frac{\mathscr{C}(\tau, \mathbf{k}) + \mathscr{C}(-\tau, \mathbf{k})}{2} \\ &= \frac{V}{2} \frac{\kappa^2}{(2\pi)^3} \int d^4x e^{i(\omega t - \mathbf{k}\mathbf{x})} \left\langle \frac{1}{2} \left\{ T_{ij}^{TT}(t, \mathbf{x}), T_{ij}^{TT}(0,0) \right\} \right\rangle_\beta , \end{aligned} \tag{4.37}$$

since translational invariance implies that $\langle T_{ij}^{TT}(-t,-\mathbf{x})T_{ij}^{TT}(0,0)\rangle_\beta = \langle T_{ij}^{TT}(0,0)T_{ij}^{TT}(t,\mathbf{x})\rangle_\beta$. Now we can finally write an expression for the energy density emission in the form of GWs $\rho_{GW} = \frac{\langle \bar{E}_{GW}\rangle_\beta}{V}$, that is

$$\frac{d\rho_{GW}}{dtd^3k} = \frac{1}{2}\frac{\kappa^2}{(2\pi)^3}\int d^4x e^{ikx}\left\langle \frac{1}{2}\left\{T_{ij}^{TT}(t,\mathbf{x}), T_{ij}^{TT}(0,0)\right\}\right\rangle_\beta . \tag{4.38}$$

The anticommutator $\{,\}$ in (4.38) is of course a convenient computational artifact, since the order of the operators does not matter at a classical level. The anticommutator is telling us that the generation of GWs is a consequence of the (symmetrized) fluctuations of the energy momentum tensor.

4.3.2 Quantum particle treatment

In order to get a quantum-particle description of the gravity coupling, we emulate the procedure followed in section 3 and we compare the result with the previous classical treatment of the GWs.

The natural candidate to represent the source for gravitational metric perturbations in the TT gauge is obviously the stress energy tensor $T^{TT\mu\nu}$. Hence, the interacting term in the Lagrangian should amount $\mathscr{L}_I \sim \kappa T^{TT\mu\nu}(x)h_{\mu\nu}^{TT}(x)$, so that κ is understood as a quantum-gravitational coupling constant. Given this picture, the particle-interpretation of metric fluctuations is straightforward. We have naturally given birth to the graviton, which couples to matter through the stress-energy tensor with strength controlled by κ. Note that since the fluctuations are GWs, they propagate at the speed of light and hence their particle-representation is massless. In this fashion, we can understand a graviton as an *spin 2 photon*. In order not to make things too dense, let us simply mimic the spin 1 case we depicted in 3.3.2, but now with an identity over polarization tensors (analog to (3.22)) that gracefully yields

$$\sum_\lambda \epsilon_{ij}^{TT(\lambda)}(k)\epsilon_{mn}^{TT(\lambda)*}(k) = \Lambda_{ijmn}(k), \tag{4.39}$$

where Λ_{ijmn} is nothing but the projector over TT-components defined in (4.20).

A straightforward application of functional differentiation on (3.26) yields the graviton self-energy. Understanding $h_{\mu\nu}(x) = J_{\mu\nu}(x)$ as source yields, after setting $J_{\mu\nu}(x) = 0$ in the final result,

$$
\begin{aligned}
\Pi^{<}_{GW}(k) &= \kappa^2 \int d^4x e^{ikx} \left\langle : T^{TT}_{ij}(0) :: T^{TT}_{ij}(x) : \right\rangle_\beta \\
&= \kappa^2 \Lambda_{ijmn}(k) \int d^4x e^{ikx} \left\langle : T^{ij}(0) :: T^{mn}(x) : \right\rangle_\beta ,
\end{aligned}
\tag{4.40}
$$

which allows us to generalise 3.3.1 and 3.3.2. In analogy with (3.25), we can write the differential rate of graviton energy density emission from a QFT approach:

$$
\frac{d\rho_{GW}}{dtd^3k} = \frac{1}{2}\frac{\Pi^{<}_{GW}(k)}{(2\pi)^3} = -\frac{f(k_0)\Lambda_{ijmn}(k)\,\mathrm{Im}\left\{\Pi^{R}_{ijmn}(k)\right\}}{(2\pi)^3}. \tag{4.41}
$$

To check the validity of (4.41), we may compare its classical limit $k_0 \ll T$ with (4.38). Let us focus on the l.h.s. equality of (4.41). The classical limit of the Bose-Einstein distribution simply implies $f(k_0) \gg 1$. With this information, one may easily prove (see (2.92) and apply it for self-energy correlators in the low energy limit) that $\Pi^{<}(k) \to \Pi^{>}(k)$, and hence in this limit the order of the operators in the correlator does not matter anymore, and the normal-ordering also looses its meaning. Thus, (4.38) is the classical limit of (4.41).

4.4 A way forward on strong coupling computations

We finally have an expression for the graviton/GWs emission rate that is completely independent of the thermal plasma dynamics. The case we are going to consider is a source of strongly coupled plasma. This is very easily said, but rather difficult to accomplish at first glance. No Dyson formula nor any other perturbative method is of any use when it comes to a $\lambda \gg 1$ coupling constant, and hence we have to be opened to explore other less conventional ways of computing strongly-coupled correlation functions. One of these ways makes use of the AdS/CFT correspondence emanating from the so-called gauge/gravity duality. It turns out (or at least it is conjectured) that a some strongly-coupled quantum theory possesses a dual classical gravity theory that allows computing the correlators on the field theory side through general relativistic computations. These fancy words will be dissected in the following chapter, where we try to devise a self-consistent introduction to this kind of computations applied to the ThFT work field. Let us without further delay dive into this fascinating corner of modern physics.

Chapter 5

The AdS/CFT duality in the context of thermal field theory

In theoretical physics, dualities are a big deal. A duality emerges from a mathematical equivalence of two theories that apparently describe different physics. In this fashion, a variety of QFTs dual to each other exist, as well as there exist string theories with a dual string theory. However, it was not until 1997 that Maldacena stated that a particular string theory was dual to a particular QFT, giving birth to the celebrated AdS/CFT conjecture [5]. The **AdS/CFT conjecture** in its original form exactly relates a type IIB superstring theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s in $AdS_5 \times \mathrm{S}_5$ with radius of curvature L and N units of $F_{(5)}$ flux on S_5 to a supersymmetric $\mathscr{N} = 4$ Yang-Mills quantum theory with gauge group $SU(N)$ and coupling constant g_{YM} through a realisation of the holographic principle. In fact, both theories are said to be *dinamically equivalent*, and there exists a mapping between the free parameters of the theories. No more than a year later, Witten proposed the equivalence of the partition functions [6], which is what we want to use.

The previous paragraph may strike as both fancy and scary as well as exciting at first glance: relating a string theory that stands as a formal candidate for describing quantum gravity to a flat-spacetime QFT is no trivial matter. Giving a detailed explanation of the exact theoretical correspondence between both sides would take us light-years away from our computational goals. If the reader is interested, the first five chapters of Ammon and Erdmenger are very instructive and detailed [15]. We will try to shed some light on the fancy looks of the first paragraph of this section so that the reader may sense the original idea behind the duality without delving too deep on rigorous mathematical background. In fact, rather than string theory, we will use another approach to explain this duality that relies on renormalization groups of QFTs. Before going forward, we will devote some time to sketch the main ideas concerning the necessary ingredients in order to understand the duality. We will not discuss details on supersymmetry nor strings in this work (we are not even going to touch the compact S_5 part of the metric space). In fact, it can be shown that stringy corrections are negligible if one takes the so-called 't Hooft strong coupling, which very roughly stands for $N \to \infty$ on the gauge theory. The only equivalence between the parameters of the two theories we are going to use is given below (admittedly *ad hoc*)

$$G_5 = \frac{\pi L^3}{2N^2}, \tag{5.1}$$

where G_5 is the five dimensional Newton constant. For quantum chormodynamics (QCD), $N = 3$.

This chapter aims to describe the most important features of CFTs and AdS spaces in the dual context. We will see how these seemingly disconnected topics end up merging into a two-faced coin. One of them is an apparently complicated QFT and the other belongs to a classical gravitational theory.

5.1 Conformal field theories

The conformal symmetry group is an extension of the Poincaré group (formed by Lorentz transformations and translations) that contains all causality-preserving transformations. In terms of a non-trivial metric, a **conformal transformation** should leave it invariant up to a positive scale factor. If we take the original metric to be flat in Lorentzian signature, conformal transformations act on Minkowski spacetime as

$$\begin{aligned} x^\mu &\to x'^\mu(x), \\ \eta_{\mu\nu} &\to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} = \Omega^{-2}(x)\eta_{\mu\nu}, \end{aligned} \tag{5.2}$$

with $\Omega(x)$ some function of the coordinates and $\eta_{\mu\nu}$ the Minkowski metric in d dimensions with $\mu, \nu = 0, 1, \dots, d-1$. For instance, it is very easy to see that for $\Omega(x) = 1$ the metric remains invariant and hence such transformations are just Poincaré transformations $x' = \Lambda x + a$. Here,

Λ represents a Lorentz transformation $\Lambda^T \eta \Lambda = \eta$ and a is a constant vector that represents the translation.

In a general case, one may consider an infinitesimal transformation with $\epsilon(x) \ll 1$ in the style of[1]

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu + \epsilon^\mu(x), \\ \eta_{\mu\nu} &\to g'(x') \sim \eta_{\mu\nu} + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x). \end{aligned} \tag{5.3}$$

We can define $\Omega(x) = e^{-\sigma(x)}$ so that for an infinitesimal transformation we get $\Omega^{-2}(x) \sim 1 + 2\sigma(x)$. Hence, comparing (5.2) with (5.3) we can write

$$\eta_{\mu\nu}\sigma(x) = \frac{1}{2}\left(\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x)\right), \tag{5.4}$$

which upon taking a trace with $\eta^{\mu\nu}$ yields $\partial^\mu \epsilon_\mu(x) = \partial \cdot \epsilon(x) = d\sigma(x)$[2]. Taking a ∂^ν-derivative in (5.4) yields the condition

$$\left[\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\right]\left(\partial \cdot \epsilon(x)\right) = 0. \tag{5.5}$$

Hence, an infinitesimal transformation is conformal if it fulfills (5.5). Obviously the case we are interested in is the $d > 2$ case. A general solution to (5.5) can be obtained by noticing that ϵ is at most of second order in x. Hence, we can write every possible combination and write

$$\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu. \tag{5.6}$$

Each of the terms in (5.6) corresponds to a kind of continuous transformation. The first two terms are immediately recognisable as the infinitesimal parameters for **translations** $\epsilon(x) = a^\mu$ and continuous **Lorentz transformations** $\epsilon^\mu(x) = \omega^\mu{}_\nu x^\nu$ (with $\omega_{\mu\nu} = -\omega_{\nu\mu}$), and together they represent **Poincaré symmetry**. The third term $\epsilon(x) = \lambda x^\mu$ is related to **dilatations**, which are also called **scale transformations**. The last two terms $\epsilon(x)^\mu = b^\mu x^2 - 2(b \cdot x)x^\mu$ are meant to be read together, and they represent **special conformal transformations**

[1]Note that in the second line of (5.3) we used that $\eta_{\mu\nu}dx^\mu dx^\nu \to \eta_{\mu\nu}\left(dx^\mu + d\epsilon^\mu\right)\left(dx^\nu + d\epsilon^\nu\right) \sim \eta_{\mu\nu}\left(dx^\mu dx^\nu + dx^\mu d\epsilon^\nu + d\epsilon^\mu dx^\nu\right) = (\eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)dx^\mu dx^\nu$.

[2]The d here is the number of dimensions that stems from taking a trace, not a differential.

(SCT), which is the other additional continuous symmetry aside from Poincaré symmetry and dilatations.

Let us specify (5.6) only for dilatations. In this case, we find $\sigma(x) = \lambda$, for an infinitesimal transformation. If λ is taken finite, we can express the transformation under a finite dilatation is

$$\begin{aligned} x^\mu &\to x'^\mu = \lambda x^\mu \\ \eta_{\mu\nu} &\to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'_\mu}\frac{\partial x^\sigma}{\partial x'_\nu}\eta_{\rho\sigma} = \lambda^{-2}\eta_{\mu\nu}. \end{aligned} \tag{5.7}$$

A theory with conformal symmetry is scale invariant. It behaves equally no matter the length or energy scale we look it at. Indeed, the first line of (5.7) is telling us that a fictitious *coordinate grid* on spacetime is being enhanced. That the physics remain invariant under such transformation is the very definition of scale invariance.

5.1.1 A word on the Lie algebra

In (5.6) we recognize the infinitesimal parameters of translations $\epsilon^\mu(x) = a^\mu$ and Lorentz transformations $\epsilon^\mu(x) = \omega^\mu{}_\nu x^\nu$ (with $\omega_{\mu\nu} = -\omega_{\nu\mu}$). The generators of translations and Lorentz transformations are momentum P_μ and the angular momentum $J_{\mu\nu}$ respectively (it can be shown that $J_{\mu\nu} = -J_{\nu\mu}$ is an antisymmetric tensor). Both sets of generators span Lie algebras. It is particularly interesting to define the Lie algebra of the generators of Lorentz transformations $J_{\mu\nu}$. Their Lie algebra is denoted as $\mathfrak{so}(d-1,1)$ and is characterised by the commutator

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}\right). \tag{5.8}$$

A Lie algebra follows from the infinitesimal expansion of an exponential map, which denotes the corresponding Lie group. In this case, the Lie group is formed by the more familiar finite Lorentz transformations $\Lambda(\omega) = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$.

In order to be able to act on fields living in a flat spacetime, Lie algebras require a certain **representation** depending on the character of the fields (scalar, vector, spinor...) they are meant to act on. A representation consists of a map between each algebra element and a matrix. If the dimensions of the Lie algebra and of the matrix group match, the representation is called faithful. A faithful representation of $\mathfrak{so}(d-1,1)$ is the matrix group $SO(d-1,1)$ consisting of all $d \times d$ matrices A that fulfill $A^T\eta A = \eta$.

Now, just as we did with Poincaré transformations, we can consider the full transformation in (5.6). Just as the Poincaré generators associate to the first two terms, there will appear two sets of generators, one for each additional type of symmetry of the conformal group. We define the generator for dilatations as D and the generators of SCTs as K_μ. One may gather every non-vanishing commutator between generators of continuous conformal transformations (regardless of their concrete macroscopic expression, which may be found in [15], chapter 3) in order to understand their Lie algebra. It is straightforward to find the conformal algebra through every pair of non-vanishing commutators between the generators $\{P_\mu, J_{\mu\nu}, D, K_\mu\}$, (it is a long list, [15], chapter 3). There is however a very simple yet ingenious trick that renders the conformal group in a much more familiar shape.

We can redefine some of the generators in order to force them to fulfill an algebra similar to the $\mathfrak{so}(d-1,1)$ algebra depicted in (5.8) but with two extra dimensions, one time-like and the other space-like. In other words, there is a way to write the conformal algebra as $\mathfrak{so}(d,2)$[3]. Imposing antisymmetry for the new generators $\bar{J}_{MM} = 0$, $\bar{J}_{MN} = -\bar{J}_{NM}$ (with $N, M = 0, ...d+1$), we only have to define

$$
\begin{aligned}
\bar{J}_{d(d+1)} = -J_{(d+1)d} = -D, \\
\bar{J}_{\mu d} = -\bar{J}_{d\mu} = \frac{1}{2}(K_\mu - P_\mu), \\
\bar{J}_{\mu(d+1)} = -\bar{J}_{(d+1)\mu} = \frac{1}{2}(K_\mu + P_\mu), \\
\bar{J}_{\mu\nu} = J_{\mu\nu}.
\end{aligned}
\tag{5.9}
$$

One may easily check that (5.8) is fulfilled for the new $\mathfrak{so}(d,2)$ generators (5.9). Hence, the conformal group in d dimensions is isomorphic to $SO(d,2)$. It can be shown that, due to antisymmetry, the number of independent generators in the $SO(d,2)$ representation is $\frac{(d+1)(d+2)}{2}$.

[3]The commutators are the same, with the metric on $\mathfrak{so}(d-1,1)$ being $\eta = \text{diag}(-1,1,\ldots,1)$, while for $\mathfrak{so}(d,2)$ the metric is $\eta = \text{diag}(-1,-1,\ldots,1,1)$, where a -1 and a $+1$ is added.

5.1.2 The scaling dimension

As mentioned before, CFTs are scale-invariant and hence they cannot depend on any dimensionful coupling such as a mass. For instance, let us take a $\xi\Phi^4$ theory in $d = 4$ with action

$$S[\Phi(x), \partial_\mu \Phi(x)] = \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi(x)\partial_\nu\Phi(x) - \frac{\xi}{4!}\Phi^4(x) \right), \tag{5.10}$$

for which $[\xi] = 1$ and hence is scale-invariant. Under a dilatation, a CFT must remain unchanged, and hence under $x^\mu \to x'^\mu = \lambda x^\mu$ we must get $S[\Phi'(x'), \partial_\mu\Phi'(x')] = S[\Phi(x), \partial_\mu\Phi(x)]$. The integral measure d^4x transforms with the Jacobian of the transformation as

$$d^dx \to d^dx' = \sqrt{-\det(\Omega^2(x)\eta)}d^dx = \Omega^d(x)d^dx = \lambda^d d^dx, \tag{5.11}$$

and the derivative transforms as

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \lambda^2 \frac{\partial}{\partial x'^\mu}\phi(x') \tag{5.12}$$

From here on it is easy to see that in order to keep (5.10) unchanged we must require that the field transforms as

$$\Phi'(x') = \lambda^{-1}\Phi(x). \tag{5.13}$$

A natural way to interpret this result is that, as the coordinate grid of the spacetime is enhanced, the field living in it scales as a negative power of the enhancement parameter λ. For a general conformal transformation, a so-called *primary field* transforms as

$$\Phi'(x') = \Omega^{-\Delta}(x)\Phi(x), \tag{5.14}$$

with Δ the **scaling dimension** of the field. For this massless quartic scalar theory, we found $\Delta = 1$ under $\Omega(x) = \lambda$.

In fact, in a CFT, every local operator will present a scaling dimension under conformal transformations. That is why in CFTs primary local operators (as well as their sources) are labeled in terms of their conformal or scaling dimension in the style of $O_\Delta(x)$.

5.1.3 Conformal generating functional

As it is done in every QFT, we can define a generating functional from which to obtain correlation functions of local operators as

$$Z_{CFT}[J_\Delta] = \int [dO_\Delta] e^{i\int d^dx O_\Delta(x) J_\Delta(x)} = \left\langle e^{i\int d^dx O_\Delta(x) J_\Delta(x)} \right\rangle_{CFT}, \tag{5.15}$$

where $[dO_\Delta]$ represents all the possible configurations of the fields that compose the local operator $O_\Delta(x)$. It can be shown that (5.15) is an exponential of another more practical generating functional $W[J_\Delta]$, which yields connected correlators when differenciated. In other words, we can write

$$Z_{CFT}[J_\Delta] = e^{iW[J_\Delta]}, \tag{5.16}$$

and hence, the connected two point function is obtained as

$$\langle O_\Delta(x_1)O_\Delta(x_2)\rangle_{CFT} = (-i)^2 \frac{\delta W[J_\Delta]}{\delta J_\Delta(x_1)\delta J_\Delta(x_2)}. \tag{5.17}$$

Conformal invariance implies that, under a conformal transformation in the style of (5.14) (on both source and operator), (5.15) remains invariant,

$$\int d^d x' O'_\Delta(x')J'_\Delta(x') = \int d^d x O_\Delta(x)J_\Delta(x) \tag{5.18}$$

Since local operators transform as (5.14)

$$O'_\Delta(x') = \Omega^{-\Delta}(x)\Phi(x) \tag{5.19}$$

and the integral measure dx transforms with the jacobian of the transformation as in (5.11) one requires that sources transform as

$$J'_\Delta(x') = \Omega(x)^{-(d-\Delta)}J_\Delta(x). \tag{5.20}$$

Hence, sources of primary operators with scaling dimension Δ are also primary operators/fields with scaling dimension $d - \Delta$.

5.2 Anti de Sitter spacetime

In general relativity (GR), a $(d+1)$-dimensional metric space is said to be isotropic and homogeneous if the metric $g_{\mu\nu}$ that defines it is invariant under translations and under $SO(1,d)$ Lorentz transformations respectively. Both kinds of transformations are said to be isometries of the metric space if they preserve distances (that is, if the metric is invariant under them). Minkowski spacetime is indeed homogeneous and isotropic since it is invariant under Poincaré transformations, which are $\frac{(d+1)(d+2)}{2}$ independent transformations. It can be shown that this is indeed the maximum number of isometries that a metric may have. A metric spacetime with $\frac{(d+1)(d+2)}{2}$ isometries is called *maximally symmetric spacetime*. Anti-de Sitter spacetime is simply a maximally symmetric spacetime with Lorentzian signature and negative, constant Ricci scalar $R < 0$[4]. Since the Ricci scalar is constant, we are able to write the Riemann tensor in therms of the Ricci scalar. A textbook computation (found on [15] chapter 2) shows that the Riemann tensor takes the form of

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}\left(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}\right), \tag{5.21}$$

and from (5.21) we can then get the Ricci tensor $R_{\mu\nu}$. If we apply all these considerations to Einstein equations in vacuum with a cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{5.22}$$

the contraction of (5.22) with $g^{\mu\nu}$ yields

$$R = \frac{2(d+1)\Lambda}{d-1}, \tag{5.23}$$

and hence AdS_{d+1} is a vacuum solution to Einstein equations with negative cosmological constant. For the sake of completeness, we cite the concrete expression of the AdS_{d+1} metric that arises from the generic procedure of finding static and isotropic metric solutions to Einstein equations, namely

$$ds^2 = -A^{-1}(r)dt^2 + A(r)dr^2 + r^2 d\Omega_{d-1}, \tag{5.24}$$

where $A(r) = 1 + \frac{r^2}{L^2}$. The coordinates that arise from (5.24) are called *global coordinates*, since they cover the whole spacetime with no horizons.

Now that we are familiar with AdS_{d+1}, we will try to properly define its boundary. In order to imagine such thing, we embed AdS_{d+1} into $d+2$-dimensional Minkowski spacetime (with two timelike components) with metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{d}(dX^i)^2 - (dX^{d+1})^2 \equiv \bar{\eta}_{MN}dX^M dX^N \tag{5.25}$$

such that AdS_{d+1} is given by the following hypersurface in $d+2$ Minkowski spacetime,

$$\bar{\eta}_{MN}X^M X^N = -L^2, \tag{5.26}$$

[4]If you think about it the curvature has to be the same at each point of the manifold in every direction due to the symmetries.

where L is the radius of curvature of AdS_{d+1}. Note that the defining hypersurface (5.26) is invariant under $SO(d,2)$ transformations, and hence the isometry group of AdS_{d+1} is indeed $SO(d,2)$. There is an idealised boundary to AdS_{d+1} which we can best understand thanks to the embedding (5.26). The boundary is of course formed by the set of points that are *far away*, or more technically, points defined by $L \ll X^M$. In this limit, the boundary of AdS_{d+1} arises, and can be written as

$$\partial AdS_{d+1} = \left\{ [X] \,:\, X \in \mathbb{R}^{d,2}, X \neq 0, \bar{\eta}_{MN} X^M X^N = 0 \right\}, \tag{5.27}$$

which is defined as embedding of the hypersurface $\bar{\eta}_{MN} X^M X^N = 0$ in $\mathbb{R}^{d,2}$. Now we could get extremely technical playing with (5.27). It is beautiful and recommended to do so. One finds that the hypersurface defines a so-called conformal compactification of d-dimensional Minkowski spacetime $\mathbb{R}^{d,2}$, which is just a fancy way of saying that the infinities of the spacetime are included as points that allow defining finite-range coordinates. If the reader is curious, again Ammon and Erdmenger [15] are very accurate. However, alternative coordinates to (5.24) render this concept much more physically manageable.

One of these coordinates are the so-called **Poincaré patch**. They cover only half of the spacetime and hence they introduce a non-physical coordinate singularity. The concrete transformation from (5.24) to the Poincaré patch can be found in many textbooks and it is a straightforward computation. The Poincaré patch for the coordinate region $r > 0$[5] can be written as

$$ds^2 = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2}\left(-dt^2 + d\mathbf{x}^2\right) = \frac{L^2}{r^2} dr^2 + \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{5.28}$$

Note that, for fixed r, the spacetime orthogonal to this direction is Minkowski with d dimensions. This is why these coordinates can also be referred as **flat slicing**. We can perform a final change of coordinates $z = \frac{L^2}{r}$, so that the metric is written as

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\mathbf{x}^2\right) = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right). \tag{5.29}$$

We can take a slice of (5.29) so that for each z there is a flat d-dimensional metric, namely

$$ds^2\big|_z = \frac{L^2}{z^2} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{5.30}$$

[5]Note that, in order to cover the whole of AdS spacetime, antoher Poincaré patch is needed for $r < 0$. Hence, the coordinate non-physical singularity is located at $r = 0$.

Setting $z \to \infty$ in (5.30) amounts to a coordinate-like singularity and hence is not very interesting. However, the limit $z \to 0$ is indeed very interesting. It is the limit in which, as we advanced in (5.27), $r \gg L$, which is inverted for the z coordinate, $z \ll L$. If we manage to render it well-behaved, the so-called conformal boundary of AdS spacetime will be located at $z = 0$. Indeed, the AdS metric (5.29) is ill-defined for $z = 0$, and so we somehow have to continue the metric to ensure finiteness. This can be accomplished by multiplying the flat metric (5.29) by a so-called *defining function*, that has to cancel z^{-2} in the $z = 0$ limit and has to be a positive and smooth function of all coordinates (z, x^μ). With these requirements, a defining function can be written as $f^2(z, x) = \frac{z^2}{L^2}\Omega^2(x)$, so that we get not a single one but a whole class of boundary metrics in terms of $f^2(z, x)$. Hence, the continued metric reads

$$ds'^2\big|_{z=0} = \Omega^2(x)\eta_{\mu\nu}dx^\mu dx^\nu. \tag{5.31}$$

There is a d-dimensional **conformal structure** living on the boundary $z = 0$ of AdS_{d+1}.

5.2.1 Field dynamics in AdS background

In order to keep the concepts clear, we will study the dynamics of a scalar field in AdS_5 with Euclidean signature. The metric of such space is given by

$$ds^2 = \frac{L^2}{z^2}\left[dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu\right], \tag{5.32}$$

and the determinant of this metric is $g = \frac{L^5}{z^5}$. The action of a massive scalar field of mass m fluctuating in AdS_5 can be written as

$$S_\phi[g, \phi] = \frac{1}{2}\int d^5x\sqrt{g}\left[g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\right]. \tag{5.33}$$

The EOMs of the field are easily obtained if one varies the action with respect to (w.r.t. from now on) the scalar field,

$$\frac{1}{\sqrt{g}}\partial_M\left[\sqrt{g}g^{MN}\partial_N\phi\right] - m^2\phi = 0. \tag{5.34}$$

Introducing the metric (5.32) yields the following equation for the field ϕ

$$\partial_z^2\phi - \frac{3}{z}\partial_z + \delta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{m^2L^2}{z^2}\phi = 0. \tag{5.35}$$

If we take the Fourier transform of ϕ on the x^μ coordinates $\phi(z, x) = \int \frac{d^4k}{(2\pi)^4}e^{ikx}\phi_k(z, k)$, we can obtain an equation for the k-th mode that reads

$$\left[\partial_z^2 - k^2 - \frac{3}{z}\partial_z - \frac{m^2L^2}{z^2}\right]\phi_k = 0. \tag{5.36}$$

We can propose a solution to (5.36) in the form of $\phi_k(z) \sim z^\alpha$ and solve near the boundary $z \sim 0$. The equation that emerges from introducing the ansatz in (5.36) is

$$\left[\alpha(\alpha-1) - 3\alpha - m^2L^2\right] z^{\alpha-2} - k^2 z^\alpha = 0. \tag{5.37}$$

Since we are considering $z \sim 0$, it is clear that the only relevant term is the $\sim z^{\alpha-2}$ one. If we equate the coefficient of the leading term to 0, we find that the allowed exponents are

$$\alpha_\pm = 2 \pm \sqrt{4 - m^2L^2}. \tag{5.38}$$

A useful redefinition is

$$\begin{aligned} \alpha_+ &\equiv \Delta, \\ \alpha_- &\equiv 4 - \Delta. \end{aligned} \tag{5.39}$$

In order not to have oscillating modes, we require $\alpha_\pm \in \mathbb{R}$, and hence we require $m^2 \geq -\frac{4}{L^2}$. If this is the case, we clearly have $4 - \Delta \leq \Delta$. This discussion will be relevant in the following, so keep it in mind. Using (5.39), we see that the near-boundary solution behaves like

$$\phi_k(k, z) \sim a(k) z^\Delta + b(k) z^{4-\Delta}, \tag{5.40}$$

which Fourier transforms as

$$\phi(x, z) \sim a(x) z^\Delta + b(x) z^{4-\Delta}. \tag{5.41}$$

A scalar field in AdS transforms like a scalar under an AdS isometry. For instance, one can easily check that under the scale transformation

$$\begin{aligned} x^\mu &\to x'^\mu = \lambda x^\mu, \\ z &\to z' = \lambda z, \end{aligned} \tag{5.42}$$

the metric (5.32) remains invariant. Hence, under the isometry (5.42), a scalar field in AdS transforms as

$$\phi(x, z) \to \phi'(x', z') = \phi(x, z). \tag{5.43}$$

If we plug the near-boundary solution (5.41) into (5.43), we can see that the functions $a(x)$ and $b(x)$ are forced to transform as

$$\begin{aligned} a'(x') &= \lambda^{-\Delta} a(x), \\ b'(x') &= \lambda^{-(4-\Delta)} b(x). \end{aligned} \tag{5.44}$$

The key here is to notice that $a(x)$ transforms like a local operator of a CFT (see e.g. (5.19)), while $b(x)$ transform like a source of a CFT (see e.g. (5.20)) under a conformal dilatation.

Moreover, if we take the dominant term in the near-boundary solution (5.41) with $z \sim 0$, we can write

$$\phi(x,z) \sim b(x) z^{4-\Delta} \implies b(x) \sim z^{-(4-\Delta)} \phi(x,z) \tag{5.45}$$

so that $b(x)$ is

$$b(x) = \lim_{z \to 0} z^{-(4-\Delta)} \phi(x,z). \tag{5.46}$$

This identities are very interesting because z can be understood as a variable scaling parameter. More technically, a CFT source $b(x)$ is related to the boundary value of an AdS field $\phi(x,z)$. The coordinate z itself is associated with a scale. But, what is this scale?

5.3 The correspondence

From the previous discussion we can conclude that a d-dimensional CFT can be put to live on the d-dimensional boundary $z = 0$ of AdS_{d+1}. Although not yet very clearly, one senses that there is an intimate relation between these two physical frameworks. In this section we finally motivate their concrete relation. Let us motivate this connection through the Wilsonian renormalization group. This discussion is inspired on some excellent notes by A. Ramallo [10].

The basic idea behind renormalization is to average microscopic UV degrees of freedom from the couplings (or sources) of a given theory by decreasing the energy scale (or increasing the length scale). This can be accomplished by considering that the spacetime where the theory lives is realized as a lattice with spacing a. In this way, in every lattice site there is an average of the physical variables. To state things clearly, for a lattice spacing a one writes the generating functional of a CFT as [6]

$$Z[J_\Delta(ax)] = \left\langle e^{i\int d^dx O_\Delta(x) J_\Delta(ax)} \right\rangle_{CFT}, \tag{5.47}$$

[6] We talk here about a lattice although the true concept would be *coarse graining*, in which the details of the theory are averaged at the scale a.

where a indicates the energy regime at which we look at the theory. Since we can *a priori* choose where to put the energy cutoff, we may interpret $z = a$ as a concrete value of an extra dimension labeled by the variable z, which represents a scale.

From renormalization theory we know that the QFT changes with the energy scale. Hence, we may consider couplings $J_\Delta(zx)$ where z corresponds to the length scale at which we look at the theory. Note that $z \to 0$ corresponds to the UV limit while $z \to \infty$ corresponds to the IR limit. Now, if the QFT is to respect conformal symmetry, we require that couplings transform under a dilatation $x^\mu \to zx^\mu$ as in (5.20). That is, we require:

$$J'_\Delta(zx) = z^{-(d-\Delta)} J_\Delta(x). \tag{5.48}$$

Let us take $z < 1$. That means that the coupling J'_Δ is the coupling at higher energies while the coupling J_Δ is the coupling at lower energies (renormalized).

Furthermore, we can take $z \to 0$ so that J'_Δ represents the UV coupling of the theory. In this case, the lattice that we were considering is not a lattice anymore since its lattice structure is now a collection of points that are truly represented by the usual x^μ. The IR coupling J_Δ still represents the lattice (renormalized) theory. Therefore, the UV coupling may be written as $J^{(0)}_\Delta(x)$, while the IR coupling is written as $J^{(z)}_\Delta(x)$, where the z remarks the fact that the IR theory is renormalized and posseses a finite lattice parameter. Finally, the UV coupling is written as

$$J^{(0)}_\Delta(x) = \lim_{z\to 0} z^{-(d-\Delta)} J^{(z)}_\Delta(x). \tag{5.49}$$

We are now on the crucial crossroads. We can see that (5.46) and (5.49) are the same expression for $d = 4$. Hence, we can associate the z-direction of AdS spacetime with a renormalization scale defining the different QFTs that live on the flat slices (see 5.1). The $z = 0$ boundary theory is a CFT whose sources are boundary values of AdS fields. Applying an isometry in AdS (5.42) amounts to a renormalization procedure in the QFT. This statement looks now trivial, but it is not. The statement is technically the following:

$$\text{AdS} : (x^\mu, z) \to (\lambda x^\mu, \lambda z) \iff \text{CFT} : x^\mu \to \lambda x^\mu + \text{Renormalization scale } z. \tag{5.50}$$

There is a matching of the symmetries of both theories when the boundary limit is taken.

Now we can get rid of every notational artifact to write a powerful statement:

$$J_\Delta(x) = \lim_{z\to 0} z^{-(4-\Delta)} \phi(x, z) = b(x), \tag{5.51}$$

which says that the source of the CFT $J_\Delta(x)$ is related to the boundary value of an AdS bulk field $\phi(x,z)$. In addition to the source, we can identify the other term $\sim a(x)z^\Delta$ in the near-boundary AdS solution (5.41) with a local operator of the CFT. Indeed, we can write

$$\langle O_\Delta(x)\rangle = \lim_{z\to 0} z^{-\Delta}\phi(x,z) = a(x). \tag{5.52}$$

Note that we have used the result (5.46) and we have extended it for $a(x)$.

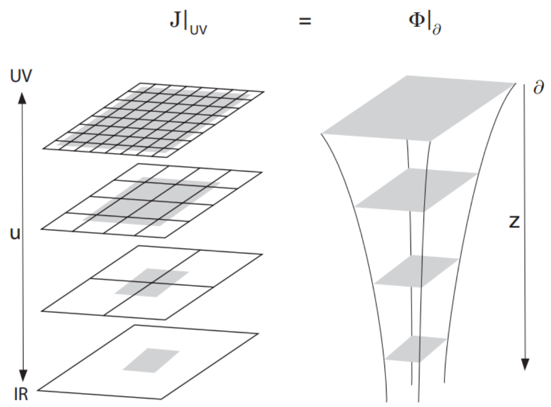

Figure 5.1: *A pictorical illustration of the correspondence. Different QFTs at different energy regimes (left) are associated with slices of the AdS space (right). Taken from* [10]

The AdS/CFT conjecture prescribes that the generating functional of a CFT is equal to the generating functional of a quantum gravity field theory in AdS (see e.g. [6]), that is

$$\begin{aligned} Z_{CFT}[J_\Delta(x)] &= \left\langle e^{i\int d^dx J_\Delta(x) O_\Delta(x)} \right\rangle_{CFT} \\ &= Z_{AdS}[\phi(z,x)]\Big|_{\lim_{z\to 0} z^{-(d-\Delta)}\phi(x,z)=J_\Delta(x)}. \end{aligned} \tag{5.53}$$

Note that $Z_{AdS} = \int \mathscr{D}\phi e^{iS_{AdS}[\phi]}$ contains quantum corrections that, in principle, cannot be neglected in Lorentzian signature. However, when classical gravity dominates, quantum corrections to the gravitational action are negligible and hence only one path is weighted (this is the definition of a classical theory). In this classical path, the AdS field is on-shell (it fulfills the equations of motion). In Euclidean signature[7], this classical limit is easily implemented,

[7]Euclidean signature is achieved through a Wick rotation $t \to -i\tau$, yielding the known relation (2.12), $S \to iS_E$. The AdS metric becomes conformally Euclidean on the boundary, with $\eta_{\mu\nu} \to \delta_{\mu\nu}$.

since small corrections on a negative real exponent vanish. Accordingly, the correspondence (5.53) continued to Euclidean signature takes the form

$$Z^E_{CFT}[J_\Delta(x)] \approx e^{-S^{on-shell}_{grav}[\phi(z,x)]}\Big|_{\lim_{z\to 0} z^{-(d-\Delta)}\phi(x,z)=J_\Delta(x)}, \tag{5.54}$$

which suggests, for on-shell particles/fields, the proposal of the following Lorentzian signature correspondence

$$Z_{CFT}[J_\Delta(x)] \approx e^{iS^{on-shell}_{grav}[\phi(z,x)]}\Big|_{\lim_{z\to 0} z^{-(d-\Delta)}\phi(x,z)=J_\Delta(x)}. \tag{5.55}$$

Lorentzian signature is crucial in order to implement causality on the correlators we want to derive from (5.55) by taking functional derivatives on the gravity side. However, there is an underlying problem we need to address.

On Euclidean signature, we only have to impose that the solution for the AdS field is regular (smooth and well-behaved) at the AdS$_{d+1}$ horizon $z \to \infty$ (in the Poincaré patch coordinates). With this condition and taking into account the value of the field at the boundary, the AdS field is uniquely determined. Indeed, this Euclidean strongly-coupled QFT is completely analog to the thermal theories we have discussed in the first part of this work. In these theories, one obtains $\sim D^<$ and $\sim D^>$ correlators, which are directly related to each other (see e.g. (2.92))[8]. This is why generally the Euclidean formalism is more easily treated.

On the other hand, there are several two-point correlators in Lorentzian signature that put causality into play (advanced, retarded, Feynman...), and hence we will need to impose an additional boundary condition on the AdS field in order to determine its behaviour. This conclusion is the heart of the mathematical discussion of Son ans Starinets [11]. Since we ultimately need the imaginary part of a retarded correlator in order to compute the emission rate of GWs from a thermal system, this is an issue that demands careful attention.

[8]Note that the theories at finite temperature we have treated are given at temperature $T = \frac{1}{\beta}$, so that if $\beta \to \infty$ the theories cease to be periodic in imaginary and hence they are given at $T = 0$, which is the case here.

5.4 Black holes and broken symmetry

Note that our field theory for thermal emission of GWs is a thermal field theory (see e.g. (4.41)). Since the theory is given for a temperature T, it is not scale invariant anymore. Hence, a basic question pops out: how do we correspond a thermal field theory with a dual gravity theory? The answer is both beautiful and shocking at first glance: the dual gravitational theory to a ThFT is AdS with a black hole! In this way, we will assign a temperature to the BH: the Hawking temperature.

Recall from chapter 2 that a bosonic ThFT is described by a partition function in periodic imaginary time with period β. Namely, $Z_{ThFT} = \mathrm{Tr}\left\{e^{-\beta\hat{H}}\right\}$ with periodicity $\tau \to \tau + \beta$. Hence, the dual Euclidean gravitational theory will also have to be periodic in Euclidean time. This is not the case of a gravity theory in pure Euclidean AdS.

Indeed, the horizon of the dual AdS gravity theory of a $T = 0$ QFT is located at $r \to 0$ (see (5.28)). Breaking the scale invariance of the theory would hence amount introducing a scale in the r direction: a new horizon $r = r_H$.

The metric of this periodic gravity theory is forced to introduce a scale. The scale is likely to be introduced on the radial coordinate (r or z), leaving the other spatial coordinates x_i invariant. Moreover, if a coordinate is rescaled, it will affect the (imaginary) time coordinate. Such metric may be obtained by generalising (5.28), yielding

$$ds^2 = g(r)\left[f(r)d\tau^2 + \delta_{ij}dx^i dx^j\right] + \frac{1}{h(r)}dr^2, \tag{5.56}$$

where, if the metric is a generalisation of AdS, $g(r) = \frac{r^2}{L^2}$ as in (5.28). Also, both $h(r)$ and $f(r)$ are meant to have a first order zero at the new horizon[9]. Hence, in the vicinity of the new horizon r_H one expands these functions as

$$\begin{aligned} f(r) &\sim f'(r_H)(r - r_H), \\ h(r) &\sim h'(r_H)(r - r_H), \end{aligned} \tag{5.57}$$

[9]Note that in the pure AdS case where the horizon is at $r = 0$, $h^{-1}(r) = \frac{L^2}{r^2}$ explodes for $r = 0$ and hence its BH generalisation must behave equally. Also, time stops running at the horizon of a BH, so it makes sense that $f(r_H) = 0$.

which from (5.56) yield a near-horizon metric

$$ds^2 \sim g(r_H)\left[f'(r_H)(r-r_H)d\tau^2 + d\mathbf{x}^2\right] + \frac{1}{h'(r_H)}\frac{dr^2}{(r-r_H)}. \tag{5.58}$$

Let us now define a new radial coordinate $\rho > 0$ such that

$$d\rho^2 = \frac{1}{h'(r_H)}\frac{dr^2}{r-r_H}. \tag{5.59}$$

Since $\rho > 0$, we can take the square root of (5.59) and integrate it in the interval $[r_H, r]$ to give the definition of ρ,

$$\rho = 2\sqrt{\frac{r-r_H}{h'(r_H)}}. \tag{5.60}$$

In analogy, we can define an angular coordinate as

$$\rho^2 d\theta^2 = g(r_H)f'(r_H)(r-r_H)d\tau^2, \tag{5.61}$$

which alongside with (5.60) yields

$$\theta = \tau\frac{1}{2}\sqrt{g(r_H)f'(r_H)h'(r_H)}. \tag{5.62}$$

Now the (τ, r) part of the near-horizon metric (5.58) is locally the metric of a plane in polar coordinates $d\rho^2 + \rho^2 d\theta^2$. Hence, in order to have a well-behaved horizon $\rho = 0$, we require that the angular coordinate θ is periodic with period 2π. Hence, this periodicity condition is the gravitational equivalent to the QFT periodicity $\tau \to \tau + \beta$. Matching both periods yields

$$2\pi = \frac{\beta}{2}\sqrt{g(r_H)f'(r_H)h'(r_H)}, \tag{5.63}$$

which gives the final identity

$$\beta = \frac{4\pi}{\sqrt{g(r_H)f'(r_H)h'(r_H)}}. \tag{5.64}$$

A temperature $T = \beta^{-1}$ is assigned to a BH. Hence, there is a correspondence between the temperature of the BH (called Hawking temperature) and the temperature of the thermal field theory.

5.5 Euclidean vs Minkowskian correlators

Since for the computation of the GW emission we need an imaginary part of a retarded correlator which is in turn related to a Wightman correlator, let us ilustrate how we intend to compute it. Consider an scalar operator coupled to a source on a CFT,

$$Z[J] = \left\langle e^{i\int d^4x O(x)J(x)}\right\rangle, \tag{5.65}$$

so that the connected two-point function is given by

$$(i)^2 \frac{\delta \log Z[J]}{\delta J(x_1)\delta J(x_2)} = \langle O(x_1)O(x_2)\rangle_\beta \,. \tag{5.66}$$

We recognise the result of (5.66): it is a Wightman correlator for a local scalar operator. We may take without loss of generality $x_1 = 0$, $x_2 \equiv x$ and define

$$G^<(x) = G^>(-x) = \langle O(0)O(x)\rangle_\beta \,, \tag{5.67}$$

where we used translational invariance. As we know, Wightman correlators are related to imaginary parts of retarded correlators (see e.g. (2.91)), namely

$$G^<(k) = -2f(k)\,\mathrm{Im}\{G^R(k)\}, \tag{5.68}$$

which is (up to indices of course) the crucial factor on the expression for the GW emission rate (4.41).

There is however a way around. Indeed, given the coupling

$$S = S_0 + \int d^4x J(x)O(x), \tag{5.69}$$

we know from linear response theory that the response of the system to the coupling is given in terms of a convolution of a Green function with the source: the retarded propagator. Hence, the value of the one point function over time is given by

$$\langle O(x)\rangle = \langle O(0)\rangle + \int d^4x G^R(x)J(x), \tag{5.70}$$

with $G^R(x)$ the retarded propagator. If we managed to impose boundary conditions compatible with causality, we would be able to interpret the result of a one-point function in terms of a retarded propagator.

The source of the field theory for the coupling of interest $T^{\mu\nu}g_{\mu\nu}$ will be dual to an spin-two field in AdS_5: the metric. The boundary value of the AdS field will be the value of the source of the field theory.

5.6 A way to correlation functions

In this section we aim to get to correlation functions through some practical cases. We begin by studying a vector field in AdS to then apply the correspondence. The results of this case will be used to motivate a more general formalism to find correlation functions.

5.6.1 Vector field in AdS

A massless vector field follows an action given by

$$S = -\int d^4x dz \frac{1}{4}\sqrt{-g} g^{MN} g^{LP} F_{ML} F_{NP}, \tag{5.71}$$

with $F_{MN} = \partial_M A_N - \partial_N A_M$ and the gauge fixing condition $A_z = 0$. We consider an AdS Minkowskian metric given by $ds^2 = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2)$, with determinant $\sqrt{-g} = \frac{L^5}{z^5}$. Varying (5.71) we get

$$\delta S = \int d^4x dz \left[\partial_M \left(\sqrt{-g}F^{ML}\right)\delta A_L - \partial_M \left(\sqrt{-g}F^{ML}\delta A_L\right)\right]. \tag{5.72}$$

The first term are the EOMs of the vector field, while the second term is a boundary term. Let us consider a Fourier mode from $A_\mu(x,z) = \int \frac{d^4q}{(2\pi)^4} e^{iqx} A^q_\mu(q,z)$. If the EOMs are satisfied (on-shell action), the z-dependence of the vector field follows from

$$\partial_z^2 A^q_\mu(q,z) - \frac{1}{z}\partial_z A^q_\mu(q,z) = 0. \tag{5.73}$$

We may propose a near-boundary solution just like we did with the scalar field, that is

$$A^q_\mu(q,z) \sim z^\alpha. \tag{5.74}$$

We find a solution that reads

$$A^q_\mu(k,z) \sim a_\mu(q) z^2 + b_\mu(q), \tag{5.75}$$

which is Fourier transformed to give

$$A_\mu(x,z) \sim a_\mu(x) z^2 + b_\mu(x). \tag{5.76}$$

Now we must require that $A_\mu(x,z)$ behaves like a vector under a diffeomorphism, that is

$$A'_\mu(x',z') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x,z). \tag{5.77}$$

Note that it is $b_\mu(x)$ the term that dominates on the boundary, and hence it will be identified with the QFT source. If that diffeomorphism is a scale transformation $(x^\mu, z) \to (\lambda z^\mu, \lambda z)$, we end up finding that a_μ and b_μ should transform as

$$\begin{aligned} a'_\mu(x') &= \lambda^{-3} a_\mu(x), \\ b'_\mu(x') &= \lambda^{-1} b_\mu(x), \end{aligned} \tag{5.78}$$

and hence they can be identified as local operator and source of a CFT. In particular, $\Delta = 3$ corresponds to a local operator of a QFT, which will be a conserved current.

We turn our attention to the second term in (5.72), the boundary term. In fact, since the action is considered on-shell, we can write

$$\delta S^{on-shell} = -\int d^4x dz \partial_M \sqrt{-g} F^{ML} \delta A_L. \tag{5.79}$$

Let us consider the $\partial_z(\dots)$-term on (5.79) (since the ∂_x ones are true boundary terms in the five dimensional space). If we use $\epsilon \to 0$ as a cutoff in the z-direction, can write

$$\delta S^{on-shell} = -\int d^4x \sqrt{-g} F^{z\mu} \delta A_\mu \bigg|_\epsilon^\infty. \tag{5.80}$$

If we take the boundary contribution, we can write

$$\delta S^{on-shell}_{bound} = \int d^4x \left[\sqrt{-g} g^{zz} g^{\mu\nu} \partial_z A_\nu(x,\epsilon)\right] \delta A_\mu(x,\epsilon). \tag{5.81}$$

Note that in the boundary $\delta A_\mu(x,\epsilon) \sim b_\mu(x)$, and hence it is the source of the CFT. If we take a look a the the correspondence, we can write

$$\log Z_{QFT} = S^{on-shell}_{bound} \implies \frac{\delta \log Z_{QFT}}{\delta b_\mu(x)} \delta b_\mu(x) = \frac{\delta S^{on-shell}_{bound}}{\delta b_\mu(x)} \delta b_\mu(x) = \delta S^{on-shell}_{bound}. \tag{5.82}$$

In other and clearer words, we have found the one point correlation function of the dual CFT local operator. There will be a term when developing (5.81) that goes like

$$\langle O^\mu(x) \rangle = \frac{\delta S^{on-shell}_{bound}}{\delta b_\mu(x)} = \sqrt{g} g^{zz} g^{\mu\nu} \partial_z \left(z^2 a_\nu(x,\epsilon)\right) \bigg|_{z=\epsilon}, \tag{5.83}$$

and that term will give the one point function of the CFT local operator. The term we will look for will be a product *something* $\times z^2 a_\mu(x) \times \delta b^\mu(x)$, where *something* $\times z^2 a_\mu(x)$ will give the one point function. Be careful, because the limit $z \to 0$ may not render the term finite, and hence the action will have to be renormalized. There is of course a way to state things more formally and much less messy.

It can be checked that

$$B^\mu(x) \equiv -\frac{\partial \mathscr{L}}{\partial\left(\partial_z A_\mu\right)}\bigg|_{z=0} = -\sqrt{-g} g^{zz} g^{\mu\nu} \partial_z A_\nu(x,z)\bigg|_{z=0}, \tag{5.84}$$

which allows writing

$$\delta S^{on-shell}_{bound} = -\int d^4x B^\mu(x) \delta b_\mu(x). \tag{5.85}$$

$B^\mu(x)$ is a sort of canonical momentum that, when renormalized, will give the one-point function, $B^\mu(x) \sim \langle O^\mu(x) \rangle$. For instance, in Minkowskian QFT we may have a generating functional

$$Z_{CFT}[A_\mu] = \left\langle e^{i \int d^4x J^\mu(x) A_\mu(x,\epsilon)} \right\rangle, \tag{5.86}$$

so that the current one point function is given by

$$(-i) \frac{\delta \log Z_{CFT}[A_\mu]}{\delta A_\mu(x_1)} = \langle J_\mu(x_1) \rangle = - \langle B_\mu(x_1) \rangle, \tag{5.87}$$

where $A_\mu(x) = a_\mu(x)$ from (5.78). Since $S^{on-shell}_{bound} \simeq Z_{CFT}$, it is clear that the source for the current correlators is the boundary value of a vector field and that the one point function for the vector current of the QFT is indeed $B^\mu(x)$.

Note that we have left aside all normalization matters. They will be treated *in situ* on the final computation.

5.6.2 Tensor field in AdS

The only tensor field that lives in AdS is of course the metric itself. Since AdS has a boundary, we will have to include the so-called Gibbons-Hawking-York (GHY) term to the action principle. Namely,

$$S = C_5 \left[\int d^4x dz \sqrt{-g}(R - 2\Lambda) + 2 \int_\partial d^4x \sqrt{-\gamma} K \right] \equiv S_\Lambda + S_{GHY}, \tag{5.88}$$

where we have a lot to explain. The line element on this spacetime reads

$$ds^2 = g^{(B)}_{MN} dx^M dx^N \equiv \frac{L^2}{z^2} dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu, \tag{5.89}$$

where the metric $\gamma_{\mu\nu}$ will define the metric at a radial slice.

The constant C_5 is the expected for a five-dimensional gravity theory, $\eta = \frac{1}{2\kappa_5^2}$, where κ_5^2 is of course related to the five-dimensional Newton constant $\kappa_5^2 = 8\pi G_5$. The first term is proportional to the Einstein-Hilbert action with a cosmological constant, which as we know is $\Lambda = -\frac{6}{L^2}$ for AdS_5 background. The coordinates are given by $x^M = (z, t, x_1, x_2, x_3)$.

The second term is a boundary term defined on the four-dimensional boundary $z = \epsilon$ of AdS_5. Indeed, the **induced metric** on the boundary is just the AdS_5 metric evaluated at $z = \epsilon$, namely

$$ds^2_{bound} = \frac{L^2}{\epsilon^2} \eta_{\mu\nu} dx^\mu dx^\nu. \tag{5.90}$$

In Euclidean signature one replaces the Minkowski metric by $\delta_{\mu\nu}$. The square root of the determinant of the induced metric is then (in Lorentzian signature)

$$\sqrt{-\gamma} = \frac{L^4}{\epsilon^4}. \tag{5.91}$$

We have to explain what K is. Technically, K is the trace of the extrinsic curvature on the boundary. The **extrinsic curvature** is defined up to a unit vector perpendicular to the boundary. In our case, this is a vector in the z-direction, n_z so that $n^z n_z = 1$, which leads to the following normal vector

$$n_z = \sqrt{g_{zz}} = \frac{1}{\sqrt{g^{zz}}}. \tag{5.92}$$

The extrinsic curvature is

$$K_{\mu\nu} = \nabla_\mu n_\nu = \partial_\mu n_\nu - \Gamma_{\mu\nu}{}^{M} n_M = -\Gamma_{\mu\nu}{}^{z} n_z, \tag{5.93}$$

where $\Gamma_{MN}{}^{L}$ are the Christoffel symbols en the five-dimensional spacetime. In our case, the Christoffel symbol appearing in (5.93) reduces to $\Gamma_{\mu\nu}{}^{z} = -\frac{1}{2} g^{zz}\partial_z g_{\mu\nu}$, and thus we can write the extrinsic curvature as

$$K_{\mu\nu} = \frac{1}{2\sqrt{g_{zz}}}\partial_z \gamma_{\mu\nu}. \tag{5.94}$$

The trace of the extrinsic curvature is just

$$K = \gamma^{\mu\nu} K_{\mu\nu} = \frac{1}{2\sqrt{g_{zz}}}\gamma^{\mu\nu}\partial_z \gamma_{\mu\nu}. \tag{5.95}$$

If we want to compute correlation functions of the field theory that lives on the boundary, we are interested in varying the action (5.88). This variation can be approached by considering small metric fluctuations to our AdS_5 background metric $g^{(B)}_{MN}$. Namely, we consider perturbations $\delta g_{MN} \equiv h_{MN}$ with $|h| \ll 1$ so that

$$g_{MN}(z) \to g_{MN}(z) + h_{MN}(x, z), \tag{5.96}$$

where we got rid of the superindex (B) for background metric. The inverse of (5.96) simply picks a minus in the h^{MN}. There is no formal difference between these perturbations and the ones we considered to study GWs. They are simply defined with one extra dimension, and in a background that is not flat, but rather it is AdS_5.

If we introduce the perturbed metric (5.96) into (5.88), we will obtain terms with no dependence on the perturbations $\sim h^0$ (background terms), and terms that depend on them up to linear order $\sim h^1$. Namely, one would get

$$S + \delta S \sim S(h^0) + \delta S(h^1), \tag{5.97}$$

so that the variation of the action is considered to first order in h. The variation of the action reads

$$\begin{aligned}\delta S = C_5 \int d^4xdz \left[\delta(\sqrt{-g})(R-2\Lambda) + \sqrt{-g}g^{MN}\delta R_{MN} + \sqrt{-g}R_{MN}\delta g^{MN})\right] \\ + 2C_5 \int d^4x \left[\delta(\sqrt{-\gamma})K + \sqrt{-\gamma}\gamma^{\mu\nu}\delta K_{\mu\nu} + \sqrt{-\gamma}K_{\mu\nu}\delta\gamma^{\mu\nu}\right].\end{aligned} \tag{5.98}$$

The terms on the first integral yield the EOMs of the system. The variation of the determinant poses no problem and yields

$$\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}g^{MN}\delta g_{MN}. \tag{5.99}$$

The variation of the metric with upper indices is also easily computed and yields

$$\delta g^{MN} = -g^{ML}g^{NP}\delta g_{PL}. \tag{5.100}$$

It is very easy to see that the combination of these two variations yields

$$\begin{aligned}\delta S = C_5 \int d^4xdz\sqrt{-g}\left[-R^{MN} + \frac{1}{2}Rg^{MN} - \Lambda g^{MN}\right]\delta g_{MN} \\ + C_5 \int d^4xdz\sqrt{-g}g_{MN}\delta R^{MN} + \delta S_{GHY}.\end{aligned} \tag{5.101}$$

We identify the Einstein tensor (up to a minus sign) on the first integral on (5.101). If the action is given on-shell, this will be equal to 0 in vacuum and will determine the dynamics of the perturbation. Einstein equations read now

$$G^{MN} = R^{MN} - \frac{1}{2}Rg^{MN} + \Lambda g^{MN} = 0. \tag{5.102}$$

These equations can be written at order h^0 to describe the background or to order h^1 to describe the perturbations in a linearised regime. If we take the first order Einstein equations from (5.102), we get

$$R^{(1)MN} - \frac{1}{2}R^{(1)}g^{MN(0)} - \frac{1}{2}R^{(0)}h^{MN} + \Lambda h^{MN} = 0, \tag{5.103}$$

which can be written as (remember that $R^{(0)} = -\frac{20}{L^2}, \Lambda = -\frac{6}{L^2}$)

$$G^{MN(1)} = R^{MN(1)} - \frac{1}{2}R^{(1)}g^{MN(0)} + \frac{4}{L^2}h^{MN} = 0. \tag{5.104}$$

From (5.104) one obtains the EOMs for the perturbation.

The variation of the Ricci tensor is however much trickier when a boundary is involved. A textbook computation yields

$$\int d^4xdz\sqrt{-g}g^{MN}\delta R_{MN} = \int d^4xdz\sqrt{-g}\nabla_P \left(g^{MN}\delta\Gamma_{MN}{}^P - g^{PN}\delta\Gamma_{MN}{}^M\right), \tag{5.105}$$

which expresses the integral of a divergence on a five-dimensional volume U with measure $dV = d^4xdz\sqrt{-g}$. We can define a four-dimensional surface normal to the z-direction ∂U with measure $dS = d^4x\sqrt{-\gamma}$. Thus, Stokes theorem allows writing

$$\int_U dV\nabla_P F^P = \int_{\partial U} dS n_z F^z. \tag{5.106}$$

A direct identification of (5.105) and (5.106) allows writing

$$\int d^4xdz\sqrt{-g}g^{MN}\delta R_{MN} = \int d^4x\sqrt{-\gamma}n_z \left(g^{MN}\delta\Gamma_{MN}{}^z - g^{zN}\delta\Gamma_{MN}{}^M\right), \tag{5.107}$$

which can be simplified by writing

$$\begin{aligned} g^{MN}\delta\Gamma_{MN}{}^z - g^{zN}\delta\Gamma_{MN}{}^M &= g^{zz}\delta\Gamma_{zz}{}^z - g^{zz}\delta\Gamma_{zz}{}^z + g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^z - g^{zz}\delta\Gamma_{\rho z}{}^\rho \\ &= g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^z - g^{zz}\delta\Gamma_{\rho z}{}^\rho. \end{aligned} \tag{5.108}$$

It is easy to get lost along this computation, so we will write the form of the variation of the on-shell action up to this point

$$\begin{aligned} \delta S^{on-shell} &= C_5 \int d^4x\sqrt{-\gamma}n_z \left(g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^z - g^{zz}\delta\Gamma_{\rho z}{}^\rho\right) \\ &+ 2C_5 \int d^4x \left[\delta(\sqrt{-\gamma})K + \sqrt{-\gamma}\gamma^{\mu\nu}\delta K_{\mu\nu} + \sqrt{-\gamma}K_{\mu\nu}\delta\gamma^{\mu\nu}\right]. \end{aligned} \tag{5.109}$$

If we vary the determinant with (5.99) and the metric with upper indices with (5.100), we can write (using $\delta\gamma_{\mu\nu} = \delta g_{\mu\nu}$)

$$\begin{aligned} \delta S^{on-shell} &= C_5 \int d^4x\sqrt{-\gamma}n_z \left(g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^z - g^{zz}\delta\Gamma_{\rho z}{}^\rho\right) \\ &+ C_5 \int d^4x\sqrt{-\gamma} \left[\gamma^{\mu\nu}K\delta g_{\mu\nu} - 2K^{\mu\nu}\delta g_{\mu\nu} + 2\gamma^{\mu\nu}\delta K_{\mu\nu}\right]. \end{aligned} \tag{5.110}$$

The only term left to vary is the one $\sim \delta K_{\mu\nu}$. We can write

$$\delta K_{\mu\nu} = -n_z\delta\Gamma_{\mu\nu}{}^z - \Gamma_{\mu\nu}{}^z\delta n_z, \tag{5.111}$$

with the variation of the normal vector

$$\delta n_z = \delta\left(g_{zz}^{-1/2}\right) = -\frac{1}{2}g_{zz}^{-3/2}\delta g_{zz}. \tag{5.112}$$

These variations entail the following variation of the on-shell action

$$\delta S^{on-shell} = C_5 \int d^4x\sqrt{-\gamma}\left(-n_z g^{\mu\nu}\delta\Gamma_{\mu\nu}{}^{z} - n_z g^{zz}\delta\Gamma_{\rho z}{}^{\rho} - K^{\mu\nu}\delta g_{\mu\nu} + g_{zz}^{-3/2}\Gamma_{\mu\nu}{}^{z}\delta g_{zz}\right)$$
$$+ C_5 \int d^4x\sqrt{-\gamma}(\gamma^{\mu\nu}K - K^{\mu\nu})\delta g_{\mu\nu}. \tag{5.113}$$

One can verify with an expansion of the Christoffel symbols and the extrinsic curvature that the first term on (5.113) cancels out (it is rather tedious and hence not shown), leading to a final, compact result

$$\delta S^{on-shell} = -C_5 \int d^4x\sqrt{-\gamma}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right)\delta g_{\mu\nu}. \tag{5.114}$$

In (5.114) there is a symmetric tensor: the so-called **Brown-York tensor**. We define it as

$$B^{\mu\nu}(x) = \sqrt{-\gamma}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right), \tag{5.115}$$

and it fulfills

$$\frac{\delta S^{on-shell}}{\delta g_{\mu\nu}(x,\epsilon)} = -C_5 B^{\mu\nu}(x). \tag{5.116}$$

From (5.116) it is quite clear that the Brown-York tensor is related to the one point function.

The boundary value of the gravitational perturbation in AdS_5 acts like a source on the boundary theory. Indeed, for a QFT with graviton we have

$$Z_{QFT}[g_{\mu\nu}] = \left\langle \exp\left\{ i\int d^4x T^{\mu\nu}(x) g_{\mu\nu}(x,\epsilon)\right\}\right\rangle, \tag{5.117}$$

where the stress-energy tensor couples to the boundary metric, which acts as its source. The one point function is then

$$(-i)\frac{\delta \log Z_{CFT}}{\delta g_{\mu\nu}(x,\epsilon)} = \langle T^{\mu\nu}(x)\rangle = -C_5 B^{\mu\nu}(x), \tag{5.118}$$

up to more subtle renormalization procedures we will cover in the next section. If we managed to impose causally compatible boundary conditions, we could identify a retarded propagator from (5.118). Indeed, from linear response theory (see (2.78)), we can write

$$\langle T^{\mu\nu}(x)\rangle = \langle T^{\mu\nu}(0)\rangle + \int d^4x G^{(R)\mu\nu\rho\sigma}(x) g_{\mu\nu}(x), \tag{5.119}$$

and *a priori* we would be able to identify a retarded propagator from (5.118) to extract its imaginary part.

As a final comment, it is of crucial importance note that since the stress-energy tensor lives on the boundary, the five-dimensional perturbations it can couple to do not have any z-index. In other words, we are not going to consider any $\delta g_{z\mu}$ perturbations. Moreover, since we will take a projection on the TT gauge (see (4.41)), we can consider the waves travelling in the x_3-direction and an steady observer (just like we did with the GWs development) to rule out any couplings with $\delta g_{t\mu}$ and $\delta g_{x_3\mu}$ perturbations. The only possible coupling we have left is with δg_{ij} perturbations with $i, j = x_1, x_2$. Recall from the TT gauge theory that the only possible couplings are hence with $\delta g_{x_1x_2} = \delta g_{x_2x_1}$ and $\delta g_{x_1x_1} = -\delta g_{x_2x_2}$ (see e.g. (4.16)). A more mathematical discussion of the coupling to tensor modes can be found in [2].

Chapter 6

Computation of the emission rate

We finally address the computation of the thermal emission rate of GWs (4.41). In order to do so, we need a dual gravity theory able to mimic the finite temperature of the field theory. We know that this can be done with AdS+BH on the gravity side. However, there are many kinds of BHs with different features, features that may have a dual interpretation on the field theory side.

We are going to consider a Reissner-Nordström (RN from now on) BH on AdS background. The horizon will be located at $r = r_H$ in Poincaré patch coordinates (5.28) and the boundary of AdS_5 at $r \to \infty$. The RN is a charged BH. Charge is introduced on the gravity theory through a gauge field $A_M(r, x)$. The boundary value of this field is associated with a source on the field theory side. Namely, if $A_\mu^{(0)}(x)$ is the boundary value of $A_\mu(r, x)$, we will have a term $\int d^4x A_\mu^{(0)}(x) J^\mu(x)$ on the generating functional of the field theory. It turns out that the gauge field for a RN only counts with one non-vanishing component $A_t(x, r) \neq 0$. Namely, we will have a boundary field theory with a generating functional

$$Z = \left\langle \exp\left\{ i \int d^4x J^t(x) A_t^{(0)}(x) \right\} \right\rangle_\beta . \tag{6.1}$$

The temporal component of the conserved current is the **charge density**, $J^t(x) \equiv \rho(x)$. It can also be interpreted as the density of number of charges (number density). The conjugate source for the number density is of course the **chemical potential**, which we call $A_t^{(0)}(x) \equiv \mu$.

The field theory is thus charged under a $U(1)$ current (which can be associated to any additive charge such as baryon number, for example), that is, it has a global symmetry. The field theory current couples to the external gauge field on the boundary. This is exactly the case we discussed in section 3.5. Hence, the dual QFT is at finite temperature because of the BH and at finite density because of the external gauge field introduced on a RN that couples to the charge density on the boundary theory. Keep in mind that these considerations are applied to the matter that forms the plasma, whose interaction is responsible for the emission of GWs.

6.1 Reissner-Nordström black hole

An action principle for the system BH+charge is given by

$$\begin{aligned} S = & \frac{1}{2\kappa_5^2}\left\{\int d^5x\sqrt{-g}(R-2\Lambda) + 2\int_\partial d^4x\sqrt{-\gamma}K\right\} \\ & -\frac{1}{4e^2}\int d^5x\sqrt{-g}g^{MN}g^{LP}F_{ML}F_{NP} \equiv S_G + S_{GHY} + S_A, \end{aligned} \tag{6.2}$$

with $\kappa_5^2 = 8\pi G_5$, where G_5 can be mapped to the field theory as is is shown in (5.1). When we vary the action w.r.t. the metric[1], we will get the contributions of the Einstein tensor from the first term, the Brown-York tensor from the first and the second, and the stress-energy tensor of the gauge field from the third term. It is easily checked that

$$\delta S_A = \frac{1}{2}\int d^5x\sqrt{-g}T_A^{MN}\delta g_{MN}, \tag{6.3}$$

with A^{MN} the stress-energy tensor of A_M defined by[2]

$$T_A^{MN} = \frac{1}{e^2}\left[g^{ML}F_{LP}F^{NP} - \frac{1}{4}F_{PL}F^{PL}g^{MN}\right]. \tag{6.4}$$

When the perturbations are entered into the stress-energy tensor, it is clear that we will have (up to linear order in the perturbations h^1)

$$T_A^{MN} = T_A^{MN(0)} + T_A^{MN(1)} + \dots \tag{6.5}$$

[1]Recall that this is done through the introduction of a perturbation to the background metric $g_{MN} \to g_{MN} + h_{MN}$.

[2]Vary the gauge-term in (6.2) and contract the different combinations of metric indices that appear.

The same thing happens for the Einstein tensor, which is expanded as

$$G^{MN} = G^{MN(0)} + G^{MN(1)} + \dots \tag{6.6}$$

Then the first order Einstein equations that describe the dynamics of the perturbations become

$$G^{MN(1)} = \kappa_5^2 T_A^{MN(1)}. \tag{6.7}$$

Thankfully, we have alreldy computed $G^{MN(1)}$ (see (5.104)) and hence we can write the Einstein equations of AdS+RN

$$R^{MN(1)} - \frac{1}{2}R^{(1)}g^{MN(0)} + \frac{4}{L^2}h^{MN} = \kappa_5^2 T_A^{MN(1)}. \tag{6.8}$$

The above equations describe the dynamics of the perturbations h_{MN}. From now on it is important that you recall that we are only interested on the dynamics of the tensor mode represented by $h_{x_1x_2}$ or more generally h_{ij}. On the other side, varying the action w.r.t. the gauge field $A_M \to A_M + \delta A_M$ yields its equations of motion. We also did this in the previous chapter and the solution is

$$\nabla_M F^{MN} = 0. \tag{6.9}$$

A solution to this Einstein-Maxwell problem is given by a metric and a gauge potential. Provided that the gauge coupling constant fulfills

$$e^2 = \frac{\kappa_5^2}{2L^2}, \tag{6.10}$$

the reader may check that the metric

$$ds^2 = \frac{L^2}{r^2 f(r)}dr^2 + \frac{r^2}{L^2}\left[-f(r)dt^2 + d\mathbf{x}^2\right] \tag{6.11}$$

with $f(r)$ a function with a pole for the BH horizon given by

$$f(r) = 1 - \frac{M}{r^4} + \frac{Q^2}{r^6} \tag{6.12}$$

and the gauge potential[3]

$$A_0(r) = \mu\left(1 - \frac{r_H^2}{r^2}\right) \tag{6.13}$$

[3] Note that the boundary value of the gauge field $A_0(r \to \infty) = \mu$ is indeed the chemical potential.

with r_H the horizon radius of the BH, constitute a solution for the problem. One also finds that the parameter μ is given by

$$\mu = \frac{\sqrt{3}}{2}\frac{Q}{L^2 r_H^2}. \tag{6.14}$$

The function $f(r)$ has to vanish for $r = r_H$, thus the relation between M and Q is

$$M = r_H^4 + \frac{Q^2}{r_H^2}. \tag{6.15}$$

This relation allows rewriting the function (6.12) as

$$f(r) = 1 - \frac{r_H^4}{r^4} - \frac{Q^2}{r_H^6}\left[\frac{r_H^4}{r^4}\left(1 - \frac{r_H^2}{r^2}\right)\right]. \tag{6.16}$$

From here we can write the Hawking temperature according to our deduction (5.64), which yields

$$T = \frac{r_H}{2\pi L^2}\left(2 - \frac{Q^2}{r_H^6}\right) \tag{6.17}$$

There is a convenient change of coordinates that can be done here. If we transform the radial coordinate as

$$u \equiv \frac{r_H^2}{r^2} \implies dr^2 = \frac{r_H^2}{4u^3}du^2, \tag{6.18}$$

and define a new parameter

$$a^2 \equiv \frac{Q^2}{r_H^6} \tag{6.19}$$

we can rewrite the function (6.16) as

$$f(u) = (1-u)(1+u-a^2u^2). \tag{6.20}$$

There is yet another useful parameter we can define to ease the story:

$$b \equiv \frac{L^2}{2r_H}. \tag{6.21}$$

Now the reader may prove that the Hawking temperature can be written as

$$T = \frac{1}{2\pi b}\left(1 - \frac{a^2}{2}\right), \tag{6.22}$$

and that all the pieces combined yield a u-coordinate metric

$$ds^2 = \frac{(\pi T L)^2}{(1-\frac{a^2}{2})u}(-f(u)dt^2 + d\mathbf{x}^2) + \frac{L^2}{4u^2 f(u)}du^2. \tag{6.23}$$

There is yet another convenient way to write the metric, which reads

$$ds^2 = \frac{L^2}{4b^2u}(-f(u)dt^2 + d\mathbf{x}^2) + \frac{L^2}{4u^2 f(u)}du^2. \tag{6.24}$$

The metric written as in (6.24) is the one we are going to use to obtain the EOMs for the perturbations $h_{xy}(u, t, \mathbf{x})$ using the first-order Einstein equations in (6.8).

As an aside note for future reference, we cite here the expression of the parameters a^2 and b in terms of the physical T and μ. One may check (see e.g. [8])

$$\begin{aligned} b^{-1} &= \frac{T}{3}\left(3\pi + \sqrt{9\pi^2 + \frac{24\mu^2}{T^2}}\right), \\ \left(1 - \frac{a^2}{2}\right) &= 2\pi b T. \end{aligned} \tag{6.25}$$

6.2 Tensor perturbations in AdS+RN

Let us now perturb the AdS+RN metric by introducing tensor perturbations in the x_1x_2-component (and in the x_2x_1 since the metric is symmetric). Since we argued we are in the TT gauge and we know that the perturbations are light-like ($k^\mu k_\mu = 0$), we can introduce the perturbations in the boundary metric as

$$ds^2 = \frac{L^2}{4b^2 u}\left(-f(u)dt^2 + d\mathbf{x}^2 + 2h_{x_1x_2}(u, t, x_3)dxdy\right). \tag{6.26}$$

We can write the perturbations as

$$h_{ij}(u, t, x_3) \equiv \int \frac{d\omega}{2\pi}\phi(u, \omega)e^{-i\omega(t-x_3)}, \tag{6.27}$$

with $i, j = x_1, x_2$.

We are now ready to tackle Einstein equations (6.7). The computation is long if done by hand, and hence we have used the software *Mathematica* in order to earn some time. The equation for $\phi(u, \omega)$ is

$$\begin{aligned} \int \frac{d\omega}{2\pi}e^{-i\omega(t-x_3)}\Bigg\{&\phi''(u, \omega) + \left[\frac{f'(u)}{f(u)} - \frac{1}{u}\right]\phi(u, \omega)' \\ &+ b^2\omega^2\frac{1 - f(u)}{uf^2(u)}\phi(u, \omega) + \frac{3a^2u^2}{4b^2}\phi(u, \omega)\Bigg\} = \kappa_5^2 T^{(1)}_{Ax_1x_2}(u, t, x_3). \end{aligned} \tag{6.28}$$

The stress-energy tensor is (we use here the required relation between couplings (6.10))

$$\begin{aligned} \kappa_5^2 T^{(1)}_{Ax_1x_2}(u, t, x_3) &= -\frac{\kappa_5^2}{2e^2}\frac{L^2}{4b^2 u}\int \frac{d\omega}{2\pi}e^{-i\omega(t-x_3)}F_{ut}F^{ut}\phi(u, \omega) \\ &= \int \frac{d\omega}{2\pi}e^{-i\omega(t-x_3)}\frac{3a^2u^2}{4b^2}\phi(u, \omega). \end{aligned} \tag{6.29}$$

Hence, the final equation for the tensor-type perturbations reads

$$\phi''(u,\omega) + \left[\frac{f'(u)}{f(u)} - \frac{1}{u}\right]\phi'(u,\omega) + b^2\omega^2\frac{1-f(u)}{uf^2(u)}\phi(u,\omega) = 0. \tag{6.30}$$

As an aside note we write the full boundary metric with upper indices

$$g^{\mu\nu}(u,k) = \frac{4b^2u}{L^2}\begin{pmatrix} -f^{-1}(u) & 0 & 0 & 0 \\ 0 & 1 & -\phi(u,\omega) & 0 \\ 0 & -\phi(u,\omega) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.31}$$

6.2.1 Near horizon behaviour

The first objective is to elucidate the near-horizon ($u \to 1$) behaviour of the perturbations. The function $f(u)$ possesses a first order pole in $u = 1$, $f(u = 1) = 0$, and so we have to be careful. In the near-horizon region (6.30) can be written as [4]

$$\phi''(u,\omega) - \frac{1}{(1-u)}\phi'(u,\omega) + \frac{b^2\omega^2}{4\left(1-\frac{a^2}{2}\right)^2}\frac{1}{(1-u)^2}\phi(u,\omega) = 0, \tag{6.32}$$

which is more easily treated if one defines

$$\mathfrak{w} \equiv \frac{b\omega}{\left(1-\frac{a^2}{2}\right)} = \frac{\omega}{2\pi T}, \tag{6.33}$$

where we used $T = \frac{1}{2\pi b}\left(1-\frac{a^2}{2}\right)$. Now equation (6.32) can be written as

$$\phi''(u,\omega) - \frac{1}{(1-u)}\phi'(u,\omega) + \frac{\mathfrak{w}^2}{4(1-u)^2}\phi(u,\omega) = 0 \tag{6.34}$$

We may propose a solution to (6.34) around the singular point $u = 1$ (known as Frobenius series) in the form of

$$\phi(u,\omega) = (1-u)^\lambda F(u), \tag{6.35}$$

where $F(u)$ is regular on $u = 1$ and can be written as a series expansion. If we plug the ansatz (6.35) into (6.34), we find that the allowed exponents are

$$\lambda_\pm = \pm i\frac{\mathfrak{w}}{2}. \tag{6.36}$$

[4] Note that around $u \sim 1$ we can write $f(u \sim 1) = (1-u)2(1-\frac{a^2}{2})$ and $f'(u=1) = -2(1-\frac{a^2}{2})$

Now we have to think for a second about the physical properties of the problem. Let us write the full solution for the perturbations (6.27) with our new findings, that is

$$h_{x_1x_2}(u,t,x_3) = \int \frac{d\omega}{2\pi} \exp\Big\{-i\omega t + i\omega x_3 \pm i\frac{\mathfrak{w}}{2} \log 1 - u\Big\}, \tag{6.37}$$

where we ommited $F(u)$ since it will be just a number on the horizon. The solution (6.37) is just a wave. Since on the horizon there is a BH, it makes physical sense that the wave cannot travel from the horizon, but rather it travels towards the horizon. Let us ignore the x_3 coordinate so that we can write

$$h_{x_1x_2}(u,t) \sim \exp\Big\{-i\omega\left[t \mp \frac{1}{4\pi T}\log(1-u)\right]\Big\}. \tag{6.38}$$

Now think about the wave itself. If we stop the clock at $t = t_1$ and take the value of $h_{x_1x_2}(u_1, t_1)$ at space point $u = u_1$, it must be that when time goes on and we stop the clock at $t = t_1 + \Delta t > t_1$, there is a point $u = u_1 + \Delta u$ (that may be greater or smaller that u_1) such that $h_{x_1,x_2}(u_1 + \Delta u, t_1 + \Delta t) = h_{xy}(u_1, t_1)$. Hence, it is true that (through a bit of algebra)

$$\mp\frac{1}{4\pi T}\log\left(1-u_1\right) = \Delta t \mp \frac{1}{4\pi T}\log\left(1-u_1-\Delta u\right). \tag{6.39}$$

If we picked the minus sign, we would require $\log\left(1-u_1-\Delta u\right) > \log\left(1-u_1\right)$ and hence we would require $\Delta u < 0$. In this case the wave would move from the horizon $u = 1$ towards the boundary $u = 0$ and thus would come out of a BH. Picking the minus sign yields the so-called **outgoing boundary condition**, and leads to an advanced Green function on the QFT side (we will see why). The reader may check that picking the plus sign yields the correct behaviour of the wave. Picking the minus sign corresponds to choosing the **ingoing boundary condition**, and it leads to a retarded Green function on the QFT side. We will of course stick to the latter boundary condition, since it is the most sensical choice.

Now that the boundary condition has been chosen, we can write the ingoing solution of the Fourier mode as

$$\phi(u,\omega) = (1-u)^{i\frac{\mathfrak{w}}{2}} F(u). \tag{6.40}$$

Do not forget that we ultimately want to find a solution near the boundary so that we can link with the QFT. The analysis on the horizon is done to rule out the outgoing solution.

6.2.2 Near boundary analysis

Near the boundary, the equation for the perturbations (6.30) can be written as

$$\phi''(u,\omega) - \frac{1}{u}\phi'(u,\omega) = 0, \tag{6.41}$$

and the ingoing solution (6.40) has to satisfy it. Since we are interested in the low frequency regime, we may propose a series solution for the function $F(u)$ in the form of

$$h_{x_1x_2}(u,\omega) = (1-u)^{i\frac{\mathfrak{w}}{2}} F(u) = (1-u)^{i\frac{\mathfrak{w}}{2}} \left(F_0(u) + i\mathfrak{w}F_1(u) + \dots\right), \tag{6.42}$$

where one could consider as many terms of the $\mathfrak{w}$-series expansion as desired (we stay at linear order). By plugging (6.42) into the equation (6.41), one finds a differential equation at order $\mathfrak{w}^0$ and another one at order $\mathfrak{w}^1$. We will give the reader the structure of the problem without displaying the long and tedious specific form of the equations, which we obtained with *Mathematica*. The first one reads

$$F_0''(u) + j(u)F_0'(u) = 0, \tag{6.43}$$

and the second one reads

$$F_1''(u) + j(u)F_1'(u) = \mathscr{G}[F_0(u), F_0'(u)], \tag{6.44}$$

where $\mathscr{G}[F_0(u), F_0'(u)]$ is a functional of the solution to the first differential equation. From (6.43) we can see that the function $F_0(u)$ is just a constant that we can set to be one, namely,

$$F_0(u) = C_0 = 1. \tag{6.45}$$

Given (6.45), the solution to (6.44) is given in terms of two constants C_1 and C_2 and reads

$$\begin{aligned} F_1(u) = C_2 + \frac{1}{4(2-a^2)} \Bigg\{ & \frac{6\,(1+2C_1)}{i\sqrt{1+4a^2}} \arctan\left[\frac{-1+2a^2u}{i\sqrt{1+4a^2}}\right] \\ & - 2\left(-1+a^2+2C_1\right)\log\left[u-1\right] + (1+2C_1)\log\left[1+u-a^2u^2\right] \Bigg\}. \end{aligned} \tag{6.46}$$

The constant C_1 is fixed by demanding regularity at the horizon. In other words, we have to kill the $\sim \log\left[u-1\right]$ term, and hence we require $C_1 = \frac{1-a}{2}$. The other constant C_2 can be fixed by demanding $F_1(0) = 0$. With this requirement, we ensure that $F(0) = 1$, and consequently $\phi(0,\omega) = 1$. Thus, we fix $C_2 = \frac{3}{2i}\frac{1}{\sqrt{1+4a^2}} \arctan\left[\frac{1}{i\sqrt{1+4a^2}}\right]$. The final solution with these nice properties is

$$\begin{aligned} F_1(u) = \frac{3}{2i}\frac{1}{\sqrt{1+4a^2}} & \left\{ \arctan\left[\frac{1}{i\sqrt{1+4a^2}}\right] - \arctan\left[\frac{1-2a^2u}{i\sqrt{1+4a^2}}\right] \right\} \\ & + \frac{1}{4}\log\left[1+u-a^2u^2\right]. \end{aligned} \tag{6.47}$$

Now, with (6.47) in our power, we can go back to the full solution (6.42) and expand it up to linear order in $\mathfrak{w}$ and near the boundary up to leading order in u, which turns out to be u^2. Thus, we can write the solution as

$$\phi(u,\omega) = 1 + \frac{i\mathfrak{w}}{2}\left(1 - \frac{a^2}{2}\right)u^2 + \dots \tag{6.48}$$

There is a very illuminating way to look at (6.48). Consider again the equation for the perturbations near the boundary (6.41). In the style of what we did in chapter 5, we consider a solution around $u = 0$ of the form $\phi(u,\omega) \sim u^\lambda$. This yields the following equation for the exponents

$$\lambda(\lambda - 1) - \lambda = 0, \tag{6.49}$$

which yields the following near-boundary behaviour:

$$\phi_k(u) \sim j(\omega, k) + v(\omega, k)u^2. \tag{6.50}$$

This is a confirmation that the solution (6.48) that we have obtained is indeed correct. The source of the boundary QFT is equal to one, as planned.

Computations of this style can be found for the other h-components can be found in [9] for BH without the $U(1)$ charge.

6.3 Brown-York tensor

Now that we have a solution, we want to compute the Brown-York tensor (5.115). We know that it is related to the one-point function (see e.g. (5.118)). We will get an structure like

$$\begin{aligned} \delta S^{on-shell} &= -\frac{1}{2\kappa_5^2}\int_\partial d^4x B^{\mu\nu}(u \to 0, x)\delta g_{\mu\nu}(u \to 0, x) \\ &= -\frac{1}{2\kappa_5^2}\int \frac{d^4k}{(2\pi)^4} B^{\mu\nu}(u \to 0, k)\delta g_{\mu\nu}(u \to 0, -k). \end{aligned} \tag{6.51}$$

However, we defined the source as $\delta g_{ij} = \frac{L^2}{4b^2u}h_{ij}$, and hence we have to extract this prefactor to write

$$\delta S^{on-shell} = -\frac{1}{2\kappa_5^2}\int \frac{d^4k}{(2\pi)^4}\frac{L^2}{4b^2u} B^{ij}(u \to 0, k)h_{ij}(u \to 0, -k). \tag{6.52}$$

Since we found $h_{ij}(0,\omega) = \phi(0,\omega) = 1$, (see (6.27) and (6.48)), the one point function is given by

$$\langle T^{ij}(k)\rangle = -\lim_{u\to 0}\frac{1}{2\kappa_5^2}\frac{L^2}{4b^2u}B^{ij}(u,k), \tag{6.53}$$

up to renormalization.

Recall that the variation of the on-shell gravity action yields

$$\delta S^{on-shell} = -\frac{1}{2\kappa_5^2}\int_\partial d^4x\sqrt{-\gamma}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right)\delta g_{\mu\nu}, \tag{6.54}$$

which has to be evaluated at the boundary $u = 0$. In our case we consider tensor-type perturbations $\delta g_{x_1x_2}$. We have to compute the Brown-York tensor appearing in (6.54). The boundary metric with upper indices was given in (6.31). The determinant of the induced metric is

$$\sqrt{-\gamma} = \frac{L^4\sqrt{f(u)}}{16b^2u^2}. \tag{6.55}$$

The extrinsic curvature can be computed from (5.94) and is given by

$$K^{\mu\nu}(u,\omega) = \begin{pmatrix} \frac{4b^2u[f(u)-uf'(u)]}{L^3f^{3/2}(u)} & 0 & 0 & 0 \\ 0 & \frac{-4b^2u\sqrt{f(u)}}{L^3} & K^{x_1x_2}(u,\omega) & 0 \\ 0 & K^{x_1x_2}(u,\omega) & \frac{-4b^2u\sqrt{f(u)}}{L^3} & 0 \\ 0 & 0 & 0 & \frac{-4b^2u\sqrt{f(u)}}{L^3} \end{pmatrix}, \tag{6.56}$$

with the off-diagonal contributions given by

$$K^{x_1x_2}(u,\omega) = \frac{4b^2u\sqrt{f(u)}}{L^3}\left[\phi(u,\omega) + u\phi'(u,\omega)\right]. \tag{6.57}$$

Now the trace of (6.56) $K = \gamma_{\mu\nu}K^{\mu\nu}$ is easily computed (remember, up to first order in h)

$$K(u,\omega) = \frac{uf'(u) - 4f(u)}{L\sqrt{f(u)}}. \tag{6.58}$$

Now we can write the Brown-York tensor $B^{\mu\nu}$ as

$$B^{\mu\nu}(u,\omega) = \begin{pmatrix} -\frac{3L}{4b^2u} & 0 & 0 & 0 \\ 0 & -\frac{L[uf'(u)-3f(u)]}{4b^2u} & B^{x_1x_2}(u,\omega) & 0 \\ 0 & B^{x_1x_2}(u,\omega) & -\frac{L[uf'(u)-3f(u)]}{4b^2u} & 0 \\ 0 & 0 & 0 & -\frac{L[uf'(u)-3f(u)]}{4b^2u} \end{pmatrix}, \tag{6.59}$$

where the off-diagonal contributions are given by

$$B^{x_1x_2}(u,\omega) = \frac{L}{4b^2u}\left[-2f(u)\phi(u,\omega) + uf'(u)\phi(u,\omega) + uf(u)\phi'(u,\omega)\right. \tag{6.60}$$

Evaluating (6.59) around $u \sim 0$[5] yields

$$B^{\mu\nu}(u\to 0,\omega) \simeq \begin{pmatrix} -\frac{3L}{4b^2u} & 0 & 0 & 0 \\ 0 & -\frac{L\left[-3+(1+a^2)u^2\right]}{4b^2u} & B^{x_1x_2}(u\to 0,\omega) & 0 \\ 0 & B^{x_1x_2}(u\to 0,\omega) & -\frac{L\left[-3+(1+a^2)u^2\right]}{4b^2u} & 0 \\ 0 & 0 & 0 & -\frac{L\left[-3+(1+a^2)u^2\right]}{4b^2u} \end{pmatrix}, \tag{6.61}$$

with

$$B^{x_1x_2}(u\to 0,\omega) = \frac{L}{4b^2u}\left[-2\phi(u,\omega)+u\phi'(u,\omega)\right]. \tag{6.62}$$

6.4 Holographic renormalization

Since the Brown-York tensor is meant to be evaluated at $u=0$, it does not take much vision to notice that (6.61) is ill defined since it presents divergent terms $\sim u^{-1}$. These terms can be removed through **holographic renormalization**, which as fancy as it may sound, is a fairly simple procedure. It consists of adding boundary terms to the action so that when we vary them they cancel the divergences present in the Brown-York tensor. These terms go like

$$\begin{aligned} S_{ren} &= S + \zeta_1 \int_\partial d^4x\sqrt{-\gamma} + \zeta_2\int_\partial d^4x\sqrt{-\gamma}R[\gamma] + \dots \\ &\equiv S + S_{ct}, \end{aligned} \tag{6.63}$$

where $\zeta_{1,2}$ are tuned to cancel the divergent terms and $R[\gamma]$ is the Ricci scalar for the boundary metric. The variation of the counterterms is

$$\delta S_{ct} = \frac{\zeta_1}{2}\int_\partial \sqrt{-\gamma}\gamma^{\mu\nu}\delta g_{\mu\nu} + \frac{\zeta_2}{2}\int_\partial d^4x\sqrt{-\gamma}\left\{-G^{\mu\nu}[\gamma]\delta g_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}[\gamma]\right\}, \tag{6.64}$$

with $G^{\mu\nu}[\gamma] = R[\gamma]^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R[\gamma]$ the boundary Einstein tensor. Note that the last term $\sim \delta R_{\mu\nu}[\gamma]$ is now a *true* boundary term (recall that it can be written as a divergence), meaning that it is a boundary term of the boundary[6], so it can be thrown away. The Einstein tensor

[5] We may again use $f(u) \sim 1-(1-a^2)u^2$.

[6] It sounds confusing, but it is just like the regular term one throws away in the usual unbounded four-dimensional GR.

term has to be put on-shell. This means that it forcefully fulfills

$$G^{\mu\nu(0)}[\gamma] + G^{\mu\nu(1)}[\gamma] + \cdots = \kappa_5^2 T_A^{\mu\nu(0)}[\gamma] + \kappa_5^2 T_A^{\mu\nu(1)}[\gamma] + \dots \tag{6.65}$$

order by order in powers of the perturbation h. Since we are evaluating the terms on the boundary, the gauge potential is a constant $A_0(u=0)=\mu$, and hence the stress energy tensor at order h^0 vanishes (see (6.4)). However, there are contributions at first order in h. The only first order non-vanishing components are of course the x_1x_2-components. Since we are in the limit $u \to 0$, we may take this limit in the first order x_1x_2 Einstein equation we computed in (6.28) to show that

$$\kappa_5^2 T^{(1)}_{Ax_1x_2}(u) = 0, \tag{6.66}$$

and hence there is no need to include a second counterterm[7].

All in all, the only constant we need to tune is ζ_1. This can be done very easily with *Mathematica*, but we show here the procedure for the component B^{xx}. The renormalized action fulfills

$$\delta S_{ren}^{on-shell} = \int_\partial d^4x \left[\frac{\zeta_1}{2}\sqrt{-\gamma}\gamma^{\mu\nu} - \frac{1}{2\kappa_5^2}B^{\mu\nu}\right]\delta g_{\mu\nu}, \tag{6.67}$$

where the integrand is finite. If we introduce the expressions, we find that

$$\frac{\zeta_1}{2}\sqrt{-\gamma}\gamma^{\mu\nu} - \frac{1}{2\kappa_5^2}B^{\mu\nu} = \frac{1}{2\kappa_5^2}\left[-\zeta_1\kappa_5^2\frac{L^2}{4bu}\frac{1}{\sqrt{f(u)}} - \frac{3L}{4bu} + \frac{(1+a^2)u}{4b}\right], \tag{6.68}$$

and by expanding $f(u) \sim 1 + \frac{1}{2}\left(1+a^2\right)u^2$, we get that the constant $\zeta_1 = -\frac{3}{L\kappa_5^2}$ removes the $\sim u^{-1}$-terms. It can be equally shown that the constant $\zeta_1 = -\frac{3}{L\kappa_5^2}$ also regularises the other components.

Now we just have to recall the one-point function (6.53) to write (bear in mind that we

[7] The contribution (6.66) does not vanish in the case $k \neq \omega$. In that case one gets a contribution that cancels the divergent terms $\sim h'_{x_1x_2}$ that appear on (6.57). You can check this at [2].

multiply by $\frac{L^2}{4b^2u}$ and we have to evaluate at the boundary)

$$\begin{aligned}
\langle T^{\mu\nu}(\omega)\rangle &= -\lim_{u\to 0}\frac{1}{2\kappa_5^2}\frac{L^2}{4b^2u}B^{\mu\nu}_{ren}(u,\omega) \\
&\simeq -\lim_{u\to 0}\frac{1}{2\kappa_5^2}\begin{pmatrix} \frac{3L^3(1+a^2)}{32b^4} & 0 & 0 & 0 \\ 0 & \frac{L^3(1+a^2)}{32b^4} & \frac{L^2}{4b^2u}B^{xy}_{ren}(u,\omega) & 0 \\ 0 & \frac{L^2}{4b^2u}B^{xy}_{ren}(u,\omega) & \frac{L^3(1+a^2)}{32b^4} & 0 \\ 0 & 0 & 0 & \frac{L^3(1+a^2)}{32b^4}\end{pmatrix},
\end{aligned} \tag{6.69}$$

with the off-diagonal components given by

$$\langle T^{x_1x_2}(\omega)\rangle \simeq -\frac{1}{2\kappa_5^2}\frac{L^3}{32b^4}\left[2\partial_u^2\phi(u,\omega)-(1+a^2)\phi(u,\omega)\right], \tag{6.70}$$

and they are written in terms of the solution (6.48) at $u=0$ as

$$\langle T^{x_1x_2}(\omega)\rangle \simeq -\frac{1}{2\kappa_5^2}\frac{L^3}{32b^4}\left[-(1+a^2)+2i\mathfrak{w}\left(1-\frac{a^2}{2}\right)\right]. \tag{6.71}$$

Indeed, the diagonal of (6.69) looks a lot like an stress-energy tensor of the field theory on a flat Minkowski spacetime with pressure and energy density defined respectively by

$$\begin{aligned} -2\kappa_5^2 P &= \frac{L^3}{32b^4}(1+a^2), \\ \rho &= 3P. \end{aligned} \tag{6.72}$$

In matrix form, the stress energy tensor can be written as

$$-2\kappa_5^2\langle T^{\mu\nu}\rangle = \frac{L^3}{32b^4}(1+a^2)\begin{pmatrix}3&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&1\end{pmatrix}+\frac{i\mathfrak{w}L^3}{16b^4}\left(1-\frac{a^2}{2}\right)\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}. \tag{6.73}$$

What (6.73) is telling us is that we have expanded the energy-momentum tensor in a series with derivatives of the source $\phi(u,\omega)$, and at the boundary $\phi(0,\omega)=1$. We could go all in with hydrodynamics [9] and we could get an exact interpretation of both terms in (6.73) in terms of energy, pressure and dissipation ($\sim \partial_u^2\phi$). We do not really have space to do so in full detail. However, there is a very nice way to look at things that connects with linear response theory we discussed around (2.78).

First of all, we have picked infalling boundary conditions and hence we expect a retarded correlator to appear. Indeed, that (6.73) has an imaginary part is already telling us that we are not going to obtain any Wightman correlator from varying it, since they are purely real. Indeed, if we had considered both the ingoing and the outgoing solution we would have ended with a sum of a quantity plus its complex conjugate, that would have given a real value.

We can interpret (6.73) from linear response theory by considering the original coupling in which

$$S = S_0 + \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x), \tag{6.74}$$

with the response of the system to the presence of the source given by the retarded propagator. In momentum space, one writes

$$\langle T^{\mu\nu}(\omega)\rangle = \langle T^{\mu\nu}(\omega = 0)\rangle + G^{(R)\mu\nu\rho\sigma}(\omega) h_{\rho\sigma}(\omega). \tag{6.75}$$

By direct identification with (6.73), we can write

$$\mathrm{Im}\Big\{G^{(R)x_1x_2,x_1x_2}(\omega)\Big\} = -\frac{1}{2\kappa_5^2}\frac{\mathfrak{w}L^3}{16b^4}\left(1-\frac{a^2}{2}\right), \tag{6.76}$$

which is the two point function we want to plug inside of (4.41)[8]. Indeed, (6.76) gives the dissipative part (remember that $\mathrm{Im}\{G^R\} \sim \rho$, the spectral function, see (2.88)). The real part would give the reactive or fluctuating part of the response (Wightman correlator difference, see (2.96)).

We have to rewrite (6.76) in terms of the physical variables T and μ with the help of (6.25). Note that we must also write the five-dimensional gravitational coupling in terms of the gauge theory parameters through (5.1) (recall that $\kappa_5^2 = 8\pi G_5$). The expansion is easily done with *Mathematica* and yields

$$\mathrm{Im}\Big\{G^{(R)x_1x_2,x_1x_2}(\omega)\Big\} \simeq -i\frac{N^2}{48\pi}\mu^2\omega T. \tag{6.77}$$

6.5 The emission rate and some final remarks

With the imaginary part of the retarded correlator (6.77) in our hands, it is time for us to compute the emission rate (4.41). We are going to study the the high density limit $\frac{\mu}{T} \gg 1$

[8] Note that if we had also considered the advanced two-point function, the final result would be real. In other words, we would have ended with some combination of Wightman propagators.

and the high temperature limit $\frac{\mu}{T} \ll 1$ both at linear order in ω. In both limits we will have to take the classical limit of the Bose-Einstein function depicted in (2.62), in which $f(\omega) \simeq \frac{T}{\omega}$.

For the high density limit, we find that the emission rate independent of the frequency and behaves like

$$\frac{d\rho_{GW}}{d^3kdt} \simeq \frac{N^2 G_N}{48\pi^3}\mu^2 T^2, \tag{6.78}$$

where $G_N = \frac{1}{m_P^2}$ and m_P is the Planck mass. As an application of (6.78), we could picture the core of a neutron star formed by quarks and gluons following the rules of QCD for which $N = 3$ and the typical values of temperature and density range around $T \sim [\mathrm{eV}, \mathrm{keV}]$ and $\mu \sim [400\mathrm{MeV}, \mathrm{GeV}]$ [4].

In the same way that we expanded the correlator (6.77) for high values of μ, we could have also done it for high temperatures. This gives the following result

$$\frac{d\rho_{GW}}{d^3kdt} \simeq \frac{3N^2 G_N}{128\pi}T^4, \tag{6.79}$$

which serves a consistency check that we have done things right as in [2].

A way forward on this work would be to embed our results in a proper neutron star model to at least estimate the strain that the radiation would cause on a detector such as LIGO. This is however left for a hopefully near future.

Bibliography

[1] B. P. Abbott *et al.* [LIGO Scientific and Virgo], *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116** (2016) no.6, 061102 doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

[2] L. Castells-Tiestos and J. Casalderrey-Solana, *Thermal emission of gravitational waves from weak to strong coupling*, JHEP **10** (2022), 049 doi:10.1007/JHEP10(2022)049 [arXiv:2202.05241 [hep-th]].

[3] C. Sivaram and K. Arun, *Thermal gravitational waves*, Open Astron. J. **4** (2011), 65-71 doi:10.2174/1874381101004010165 [arXiv:0708.3343 [astro-ph]].

[4] C. Hoyos, N. Jokela and A. Vuorinen, *Holographic approach to compact stars and their binary mergers*, Prog. Part. Nucl. Phys. **126** (2022), 103972 doi:10.1016/j.ppnp.2022.103972 [arXiv:2112.08422 [hep-th]].

[5] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231-252 doi:10.4310/ATMP.1998.v2.n2.a1 [arXiv:hep-th/9711200 [hep-th]].

[6] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998), 253-291 doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

[7] P. K. Kovtun and A. O. Starinets, *Quasinormal modes and holography*, Phys. Rev. D **72** (2005), 086009 doi:10.1103/PhysRevD.72.086009 [arXiv:hep-th/0506184 [hep-th]].

[8] C. Hoyos, N. Jokela, D. Rodríguez Fernández and A. Vuorinen, *Breaking the sound barrier in AdS/CFT*, Phys. Rev. D **94** (2016) no.10, 106008 doi:10.1103/PhysRevD.94.106008 [arXiv:1609.03480 [hep-th]].

[9] G. Policastro, D. T. Son and A. O. Starinets, *From AdS / CFT correspondence to hydrodynamics*, JHEP **09** (2002), 043 doi:10.1088/1126-6708/2002/09/043 [arXiv:hep-th/0205052 [hep-th]].

[10] A. V. Ramallo, *Introduction to the AdS/CFT correspondence*, Springer Proc. Phys. **161** (2015), 411-474 doi:10.1007/978-3-319-12238-0_10 [arXiv:1310.4319 [hep-th]].

[11] D. T. Son and A. O. Starinets, *Minkowski space correlators in AdS / CFT correspondence: Recipe and applications*, JHEP **09** (2002), 042 doi:10.1088/1126-6708/2002/09/042 [arXiv:hep-th/0205051 [hep-th]].

[12] X. H. Ge, Y. Matsuo, F. W. Shu, S. J. Sin and T. Tsukioka, *Density Dependence of Transport Coefficients from Holographic Hydrodynamics*, Prog. Theor. Phys. **120** (2008), 833-863 doi:10.1143/PTP.120.833 [arXiv:0806.4460 [hep-th]].

[13] Le Bellac M., *Thermal Field Theory*. Cambridge: Cambridge University Press; 1996.

[14] C. W. Misner, K. S. Thorne and J. A. Wheeler, W. H. Freeman, *Gravitation*, 1973, ISBN 978-0-7167-0344-0, 978-0-691-17779-3.

[15] Ammon M, Erdmenger J., *Gauge/Gravity Duality: Foundations and Applications*. Cambridge: Cambridge University Press.