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$\mathcal{N} = 2$ super-EYM coloured black holes from defective Lax matrices

Patrick Meessen[a] **and Tomás Ortín**[b]

[a] *HEP Theory Group, Departamento de Física, Universidad de Oviedo, Avda. Calvo Sotelo s/n, E-33007 Oviedo, Spain*

[b] *Instituto de Física Teórica UAM/CSIC, C/ Nicolás Cabrera 13–15, C.U. Cantoblanco, E-28049 Madrid, Spain*

E-mail: meessenpatrick@uniovi.es, Tomas.Ortin@csic.es

Abstract: We construct analytical supersymmetric coloured black hole solutions, i.e. non-Abelian black hole solutions that have no asymptotic non-Abelian charge but do have non-Abelian charges on the horizon that contribute to the Bekenstein-Hawking entropy, to two SU(3)-gauged $\mathcal{N} = 2$ $d =$ supergravities. The analytical construction is made possible due to the fact that the main ingredient is the Bogomol'nyi equation, which under the assumption of spherical symmetry admits a Lax pair formulation. The Lax matrix needed for the coloured black holes must be defective which, even though it is the non-generic and less studied case, is a minor hindrance.

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Contents

1 Introduction

In refs. [1, 2] the structure of the supersymmetric solutions to $\mathcal{N} = 2, d = 4$ supergravity coupled to vector multiplets where a subset of the isometries of the scalar manifold is gauged [3, 4], a theory we shall refer to as $\mathcal{N} = 2, d = 4$ Super-Einstein-Yang-Mills, was uncovered. As far as said characterisation is concerned, the resulting structure is *grosso modo* the same as in the ungauged case, see e.g. refs. [5, 6], but for two details: the seed functions $\mathcal{I}^\Lambda, \mathcal{I}_\Lambda$ are in general not harmonic functions on $\mathbb{R}^3$. More to the point, some of the seed functions (the *magnetic* ones $\mathcal{I}^\Lambda$ that have indices Λ in the non-Abelian gauge group and are denoted by Φ) must satisfy the non-Abelian[1] Bogomol'nyi equation on $\mathbb{R}^3$, i.e.

$$\star\,\mathsf{F} = \mathsf{D}\Phi\,, \tag{1.1}$$

whereas the other half (the *electric* ones $\mathcal{I}_\Lambda$) must satisfy an involved equation whose explicit form depends on the knowledge of the solutions to eq. (1.1).

Some supersymmetric, analytical spherically symmetric solutions to various models were constructed and analysed in refs. [1, 2, 7] and, recently, some multi-object solutions

[1]In the ungauged case (and in the ungauged directions) the seed functions also have to obey the Bogomol'nyi equation, but in its Abelian version. A solution (gauge field) is guaranteed to exist for any choice of harmonic seed functions and, for this reason, the presence of the Abelian Bogomol'nyi equation is seldom mentioned in the literature. In the non-Abelian case, however, one cannot simply give a set of seed functions satifying certain conditions: it is necessary to provide the accompanying gauge field to determine completely the solution.

were presented in ref. [8]. Apart from the globally regular solutions describing 't Hooft-Polyakov monopoles in supergravity, the most elusive and intriguing solutions in those references describe an SU(2) black hole whose non-Abelian gauge fields have no asymptotic colour charge and the physical scalars have no asymptotic v.e.v. The near-horizon solution, however, is completely specified in terms of the colour charge of the gauge fields on the horizon, which does not vanish and contributes to the Bekenstein-Hawking entropy. This behaviour is reminiscent of the Bartnik-McKinnon particle solution to SU(2) Einstein-Yang-Mills (EYM) theory [9], which is a discrete family of spherically symmetric, globally regular solutions with a YM connection that has no asymptotic colour charge. The Bartnik-McKinnon solutions were generalised to black-hole solutions in refs. [10–12] and they are given the, perhaps counter-intuitive, name of *coloured black holes* and, accordingly, the elusive solution mentioned above is called an $\mathcal{N}=2$ *Super-EYM (SEYM) coloured black hole*.

Observe that the Bartnik-McKinnon particle and their generalisations are only known numerically, and analytical examples are hard to come by. As far as we know, apart from the sugra coloured black hole of refs. [2, 7], the only other analytical coloured black hole was recently constructed by Fan & Lü [13] and is a solution to conformal gravity coupled to SU(2) YM. This last coloured black hole, like the coloured black holes of EYM, is "pure" in the sense that for their existence no other gauge field needs to be turned on. This is in contradistinction to the coloured black hole in supergravity where the regularity of the event horizon demands an additional Abelian gauge field to be active. Even so, it is, within the class of theories and (supersymmetric) solutions considered, the closest one can get to a coloured black hole.

Seeing the interesting properties of the coloured black holes, however, it would be interesting to have more analytical examples; in the $\mathcal{N}=2, d=4$ case we are quite fortunate as the main ingredient of their construction is the Bogomol'nyi equation and as was shown by Leznov & Saveliev in ref. [14], the SU(N) Bogomol'nyi equation, under the assumption of (so-called maximal) spherical symmetry, is an integrable system related to the SU(N) Toda molecule. The general solution of the SU(N) Toda molecule is known [15–18] and can be derived from a Lax pair representation for the relevant equations, with a Lax matrix that is assumed to be diagonalisable. As we will see, the solutions to the spherically symmetric Bogomol'nyi equations that are needed to construct coloured black holes are associated to a Lax matrix that is defective or, said differently, manifestly non-diagonalisable. The ones we are after can be obtained from the generic solutions as limiting cases (see e.g. ref. [19]) or by the integration algorithms presented in refs. [20, 21] and [22]. Here we will adapt Koikawa's derivation of the solution [15] to the case of a defective Lax matrix: this is done by relating the Lax pair to the standard Lax evolution equation on the generalised eigenvectors of the defective Lax matrix. The generalised eigenvectors are the vectors with respect to which the Lax matrix takes on the Jordan block form and are natural objects to use when the Lax matrix is defective. For the small N SU(N) solutions we are going to construct (SU(2) and SU(3)) this gives a convenient and easy to understand way of constructing the solutions, clarifying the general procedure of refs. [20, 21] and [22].

Once we have constructed the relevant solutions of the SU(N) Bogomol'nyi equations, we will use them to construct *bona fide* coloured black hole solutions to $\mathcal{N}=2, d=4$

SEYM theories. In this case we will limit ourselves to the construction of coloured black holes in the $\overline{\mathbb{CP}}^8$ and the $\mathbb{C}$-magic models, both of which allow for an SU(3) gauging. In both cases, we will see that, as in the SU(2) coloured black holes presented in refs. [2, 7, 8], there is no major obstruction in their construction. In fact, it should be more or less clear that coloured black-hole solutions abound in $\mathcal{N}=2, d=4$ SEYM theories.

The outline of this article is as follows: in section 2 we shall discuss the form of the gauge connections and Higgs fields compatible with spherical symmetry, relate the SU(N) Bogomol'nyi equation under the assumption of spherical symmetry to the SU(N) Toda molecule and give the Lax pair. We will then discuss the appropriate boundary conditions for the coloured solutions we are interested in and argue that they correspond to a defective Lax matrix, i.e. to a non-diagonalisable matrix. In section 3 we will discuss the general mechanism needed for extracting the solutions and apply it to the cases of SU(2) and SU(3). In section 4.2 we will then embed the SU(3) solution in the SU(3)-gauged $\overline{\mathbb{CP}}^8$ and $\mathbb{C}$-magic models and analyse the gravitational solution. Section 5 contains some conclusions and comments.

2 Spherically symmetric solutions to the Bogomol'nyi equations

The derivation is best done using Hermitean generators, which means that we use the definitions

$$\mathsf{D}\Phi = d\Phi - i\,[\mathsf{A},\Phi] \quad , \quad \mathsf{F} = d\mathsf{A} - i\;\mathsf{A}\wedge\mathsf{A}\,, \tag{2.1}$$

where A and Φ are in $\mathfrak{su}(n+1)$'s fundamental representation; for convenience we have taken the coupling constant to be one.

In the radial gauge, i.e. in the gauge where the radial component of the gauge connection vanishes, the form of the fields compatible with maximal spherical symmetry are given by (see e.g. refs. [23–26])

$$\Phi = \frac{1}{2}\,\mathrm{diag}\left(\phi_1(r)\,,\ \phi_2(r)-\phi_1(r)\,,\ \ldots\,,\ \phi_n(r)-\phi_{n-1}(r)\,,\ -\phi_n(r)\right)\ , \tag{2.2}$$

$$\mathsf{A} = J_3\ \cos(\theta)d\varphi + \frac{i}{2}\left[C - C^\dagger\right]\ d\theta + \frac{1}{2}\left[C + C^\dagger\right]\ \sin(\theta)d\varphi\,, \tag{2.3}$$

where

$$J_3 = \frac{1}{2}\mathrm{diag}(n, n-2, \ldots, 2-n, -n)\,, \tag{2.4}$$

corresponds to a spin $n/2$ irrep of $\mathfrak{su}(2)$ and is the maximal embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(n+1)$; C is the real upper-triangular matrix

$$C = \begin{pmatrix} 0 & a_1(r) & 0 & \cdots & 0 \\ 0 & 0 & a_2(r) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1}(r) & 0 \\ 0 & 0 & \cdots & 0 & a_n(r) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\,. \tag{2.5}$$

A small calculation using the fact that $[J_3, C] = C$, then shows that the Bogomol'nyi equation (1.1) reduces to the following system

$$\begin{aligned} \partial_r C &= [\ \Phi\ ,\ C\ ]\ , && (2.6)\\ 2r^2\partial_r\Phi &= \Big[\ C\ ,\ C^\dagger\ \Big]\ - 2J_3\ . && (2.7)\end{aligned}$$

The starting point of Koikawa's construction are eqs. (2.6) and (2.7). First, one redefines[2]

$$\Phi = \Psi + J_3/r \quad , \quad C = rF\ , \tag{2.8}$$

turning eqs. (2.6) and (2.7) into

$$\begin{aligned} \dot{F} &= [\ \Psi\ ,\ F\ ]\ , && (2.9)\\ 2\dot{\Psi} &= [\ F\ ,\ F^T\ ]\ , && (2.10)\end{aligned}$$

where clearly $\Psi^T = \Psi = \Psi^\dagger$ and an overdot means derivative with respect to r.

After this redefinition, the link to the Toda molecule can be easily established [17]: in terms of the components ψ_i of Ψ and f_i of F, the above equations read

$$\dot{\psi}_i = f_i^2 \quad , \quad 2\dot{f}_i = f_i\ A_{ij}\psi_j\ , \tag{2.11}$$

where A is $\mathfrak{su}(n+1)$'s Cartan matrix. Defining the new variables $T_i(r)$ by

$$f_i = \exp\left(-\frac{1}{2}\ A_{ij}T_j\right)\ , \tag{2.12}$$

we see from the f-equations that $\psi_i = -\dot{T}_i$. Upon substitution into the ψ-equation we find that the T_i variables satisfy the equations

$$\ddot{T}_i = -\exp\left(-A_{ij}T_j\right)\ , \tag{2.13}$$

which are known as the equations of motion of the $\mathrm{SU}(n+1)$ Toda molecule.

The second step in Koikawa's construction [15] is the definition of the new objects L, M from Ψ and F, which form a Lax pair, i.e.

$$\left.\begin{aligned} \mathsf{L}\ &\equiv \Psi + \tfrac{i}{2}\left(F + F^T\right)\\ \mathsf{M} &\equiv \tfrac{i}{2}\left(F^T - F\right)\end{aligned}\right\}\quad \longrightarrow \quad \dot{\mathsf{L}} = [\ \mathsf{M}\ ,\ \mathsf{L}\ ]\ , \tag{2.14}$$

The existence of a Lax pair immediately implies that the quantities ("charges") $\mathcal{C}_{(k)} \equiv \mathrm{Tr}\left(\mathsf{L}^k\right)$ are constants of motion. Observe that even though L is symmetric and complex, whereas M is purely imaginary and anti-symmetric, the conserved charges $\mathcal{C}_{(k)}$ are in fact real functions because L and M satisfy the relations [25]

$$\mathsf{L}^* = e^{i\pi J_3}\ \mathsf{L}\ e^{-i\pi J_3} \quad , \quad \mathsf{M}^* = e^{i\pi J_3}\ \mathsf{M}\ e^{-i\pi J_3}\ . \tag{2.15}$$

[2] F should not be confused with the gauge field strength F.

We are interested in Higgs fields that behave asymptotically as

$$\Phi \;\sim\; \Phi_\infty + \frac{\mathrm{P}}{r} + \mathcal{O}(r^{-2}) \;, \tag{2.16}$$

where P is the colour charge and one can take $[\Phi_\infty, \mathrm{P}] = 0$.[3] According to eq. (2.10), in order for the above asymptotic behaviour to be possible we must have[4]

$$F \;\sim\; \frac{\mathrm{S}}{r} \qquad \text{with} \quad \left[\mathrm{S}, \mathrm{S}^T\right] = 2\,(\; J_3 - \mathrm{P}\;)\;. \tag{2.17}$$

Eq. (2.9) then implies that

$$[\;\Phi_\infty\,,\;\mathrm{S}\;] = 0 \quad \text{ and } \quad [\;\mathrm{P}\,,\;\mathrm{S}\;] = 0\;. \tag{2.18}$$

Defining then a coloured solution to the Bogomol'nui equation as one for which the colour charge is zero, i.e. $\mathrm{P} = 0$, we see from eq. (2.17) that $s_i^2 = i(n+1-i)$, so that S has no vanishing entries; this immediately implies by virtue of eq. (2.18) that $\Phi_\infty = 0$. The conclusion is that if, in the maximally spherically symmetric Ansatz, we want to describe coloured solutions, defined as solutions having $\mathrm{P} = 0$, then we must look at the class of solutions with $\Phi_\infty = 0$. Observe, however, that it is possible to have solutions with $\Phi_\infty = 0$ but $\mathrm{P} \neq 0$.

For the conserved charges $\mathcal{C}_{(k)}$ we see that

$$\mathcal{C}_{(k)} = \lim_{r\to\infty} \mathrm{Tr}\left(\mathsf{L}^k\right) = \mathrm{Tr}\left(\Phi_\infty^k\right) \quad \xrightarrow{\text{coloured solutions}} \quad \mathcal{C}_{(k)} = 0\;. \tag{2.19}$$

Even though L is a complex symmetric matrix it allows for eigenvalues and if we take into account the fact that the characteristic polynomial $\det(\mathsf{L} - \lambda\,\mathrm{Id})$ can be expanded as a polynomial in λ and the $\mathcal{C}_{(k)}$, we reach the conclusion that a coloured solution is such that all eigenvalues are zero: this is only possible for non-trivial solutions if L is not diagonalisable, whence an L corresponding to a coloured solution must be a defective matrix, i.e. one that can be brought by an r-dependent similarity transformation to a Jordan block form with only zero eigenvalues.

Observe that for our purposes we also might want to impose the condition that around $r = 0$[5] we want the solution to be Coulombic, meaning that

$$\lim_{r\to 0} \Phi \;\sim\; \frac{\mathrm{Q}}{r} + \Phi_0 + \mathcal{O}(r)\;, \tag{2.20}$$

as any higher order singularity would (probably) be uncompensable in an $\mathcal{N} = 2$ supergravity setting. The solutions that we have constructed, however, automatically have this behaviour.

[3] We can always perform a constant gauge transformation to diagonalise Φ_∞ and an r-dependent one to diagonalise P.

[4] In components this reads $f_i \sim s_i/r$ for some constants s_i.

[5] This is the location of the would-be event horizon in the full gravitational solutions built from these solutions of the Bogomol'nyi equation.

3 Coloured solutions from defective Lax matrices

The Lax equation (2.14) is the integrability condition of the so-called Lax equations[6]

$$\mathsf{L}\vec{v} = \lambda\,\vec{v}\,, \tag{3.1}$$

$$\dot{\vec{v}} = \mathsf{M}\,\vec{v}\,, \tag{3.2}$$

where λ is r-independent. As was said before, the matrix L is symmetric but complex and as such it need not be diagonalisable. But since this possibility is not excluded either, let us suppose first that all the eigenvalues[7] of L are real and different [15]. This means that there are n eigenvalues λ_i $(i = 1,\ldots,n)$ and n corresponding eigenvectors $\vec{v}_i$, which due to eq. (3.2) can be taken to be orthonormal, $\vec{v}_i\cdot\vec{v}_j = \delta_{ij}$, and complete $\mathrm{Id} = \sum_i\ \vec{v}_i\vec{v}_i^{\,T}$. We can then immediately write down the spectral decomposition of L

$$\mathsf{L} = \sum_i\ \lambda_i\ \vec{v}_i\vec{v}_i^{\,T}\ , \tag{3.3}$$

and express the resolvent of L as

$$\mathsf{R}(\mu)\ \equiv\ (\mathsf{L}-\mu\ \mathrm{Id})^{-1} = \sum_i\ \frac{\vec{v}_i\vec{v}_i^{\,T}}{\lambda_i-\mu}\,. \tag{3.4}$$

The Lax equations can be turned into n first order differential equations by projection. Let us consider the SU(2) case with a basis $\{\vec{e}_1,\vec{e}_2\}$, define $x_i\equiv\vec{e}_1\!\cdot\!\vec{v}_i$ and use the definitions of L and M eqs. (2.14) to see that $\mathsf{M}_{12} = -\mathsf{L}_{12}$. Whence, from eq. (3.2), we have that $\dot{x}_i = \mathsf{M}_{12}\,\vec{e}_2\!\cdot\!\vec{v}_i = -\mathsf{L}_{12}\,\vec{e}_2\!\cdot\!\vec{v}_i$. Using these expressions in the projection of eq. (3.1) onto $\vec{e}_1$ we find

$$\dot{x}_i = \mathsf{L}_{11}x_i - \lambda_i\ x_i \qquad \text{with}\ \ \mathsf{L}_{11}\ =\ \sum_j\ \lambda_j x_j^2\,. \tag{3.5}$$

By substituting $x_i\ =\ e^{-\lambda_i r}\ a_i\ u(r)$, where a_i is some constant, we see that the general solution to eqs. (3.5) is given by

$$\frac{1}{u^2} = \aleph + \sum_j\ a_j^2\ e^{-2\lambda_j r}\quad \xrightarrow{\text{completeness}}\quad \frac{1}{u^2} = \sum_j\ a_j^2\ e^{-2\lambda_j r}\ , \tag{3.6}$$

where the completeness alluded to in the equation follows from the projection onto the 11-direction of the completeness relation of the $\vec{v}_i$. Up to this point we can, by using eqs. (2.4) and (2.14), deduce that

$$\psi_1 = 2\mathsf{L}_{11} = 2\frac{\sum_i\ \lambda_i a_i^2\ e^{-2\lambda_i r}}{\sum_j\ a_j^2\ e^{-2\lambda_j r}}\,. \tag{3.7}$$

Even though it may seem that in order to construct the full solution one would have to find all the components of the eigenvectors, this is not really needed: we can calculate the 11-component of the resolvent, $\mathsf{R}_{11}(\mu)$, in two different ways, namely by projection of

[6]The vector $\vec{v}$ is r-dependent, but we refrain from writing $\vec{v}(r)$ for the ease of reading.

[7]As we are dealing with the SU$(n\!+\!1)$ Bogomol'nyi equation, the sum of all the eigenvalues must be zero.

the spectral representation in eq. (3.4) and also by using Cramer's rule to invert the matrix $\mathsf{L}-\mu\mathrm{Id}$, given eq. (2.8). The comparison of these two expressions is enough to fix all the components. We refer the interested reader to Koikawa's article [15] for the final result. As the reader will see, using the resolvent to fix the complete solution is especially easy for the coloured solutions, at least for the low-n cases we are interested in.

In the case that the eigenvalues are degenerate we must distinguish between two cases: the non-defective case and the defective case. In the non-defective case in which L can still be diagonalised even though the eigenvalues are degenerate, a complete set of eigenvectors does still exist and the above method can be applied.

In the defective case in which the matrix can be brought to a Jordan block form but not a diagonal form, we need to introduce the so-called *generalised eigenvectors*: suppose we are given an eigenvalue $\lambda_\star$ which corresponds to some $k\times k$ Jordan block;[8] in this case there is only one eigenvector $\vec{v}_\star$ such that

$$\mathsf{L}\vec{v}_\star = \lambda_\star\vec{v}_\star \,. \tag{3.8}$$

We can, however, construct a basis for the complete eigenspace with *generalised eigenvectors* $\vec{w}(m)$ where $m=1,\dots,k$ and $\vec{w}(k)\equiv\vec{v}_\star$ defined by the property

$$\vec{w}(m+1) = (\mathsf{L}-\lambda_\star\mathrm{Id})\;\vec{w}(m) \;\equiv\; \mathsf{N}\vec{w}(m)\,, \tag{3.9}$$

with the understanding that $\vec{w}(k+1)=0$. This equation makes sense as the matrix N restricted to the generalised eigenspace corresponding to the eigenvalue $\lambda_\star$ is nil-potent of degree k, i.e.

$$\mathsf{N}^k = 0\,. \tag{3.10}$$

The integrability condition of the evolution equation

$$\dot{\vec{w}}(m) = \mathsf{M}\vec{w}(m)\,, \tag{3.11}$$

with the definitions (3.9) is nothing more than the Lax pair equation (2.14). Said differently: if we find a solution to the system formed by eqs. (3.9) and (3.11), we automatically obtain a solution to the original equations (2.6) and (2.7).

The basis $\{\vec{\omega}(m)\}$ is by construction complete but the details vary with respect to the non-degenerate case as, for example, $\vec{w}(k)\cdot\vec{w}(k)=0$,[9] so that the relevant spectral representations will not be as simple as in eq. (3.3) and (3.4). First of all, since M is antisymmetric the inner products of the generalised eigenvectors are r-independent, i.e.

$$\partial_r\left[\vec{w}(m)\cdot\vec{w}(n)\right]=0\,,\quad \text{whence}\quad \vec{w}(m)\cdot\vec{w}(n) = \left.\vec{w}(m)\cdot\vec{w}(n)\right|_{r=0}\,. \tag{3.12}$$

Second, one finds that $\vec{w}(m)\cdot\vec{w}(n) = \vec{w}(m-1)\cdot\vec{w}(n+1)$ so this product only depends on the sum of the indices $m+n$. This, together with $\mathsf{N}\vec{w}(k)=0$, implies that

$$\vec{w}(m)\cdot\vec{w}(n)=0 \qquad \text{if } m+n>k+1. \tag{3.13}$$

[8]The generalisation to more Jordan blocks, even with the same eigenvalue, is straightforward, but we refrain from describing it here in order not to clutter the equations with too many indices.

[9]The $\vec{w}(m)$ are complex.

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In fact, since eq. (3.9) determines $\vec{w}(m)$ $(m \neq k)$ up to terms proportional to $\vec{w}(k)$ we can arrange things such that

$$\vec{w}(m) \cdot \vec{w}(n) = \begin{cases} 0 : & m+n \neq k+1 \,, \\ 1 : & m+n = k+1 \,. \end{cases} \tag{3.14}$$

Having deduced the relevant inner products of the generalised eigenvectors one can write down the completeness relation on this eigenspace as

$$\mathrm{Id}|_{\lambda_\star} = \sum_{m=1}^{k} \vec{w}(k+1-m)\vec{w}(m)^T \,, \tag{3.15}$$

and the (generalised) spectral representation of L reads

$$\mathsf{L}|_{\lambda_\star} = \lambda_\star \sum_{m=1}^{k} \vec{w}(k+1-m)\vec{w}(m)^T + \sum_{m=2}^{k} \vec{w}(k+2-m)\vec{w}(m)^T \,. \tag{3.16}$$

Finally, the restriction to this eigenspace of the resolvent $\mathsf{R}(\mu)$ is

$$\mathsf{R}(\mu)|_{\lambda_\star} = \sum_{n=1}^{k} \frac{(-1)^{n+1}}{(\lambda_\star - \mu)^n} \sum_{m=n}^{k} \vec{w}(k+n-m)\vec{w}(m)^T \,, \tag{3.17}$$

Having set up the relevant algebraic structures, the deduction of the solutions follows the same route as outlined above.

3.1 SU(2)'s coloured solution revisited

Given that the sum of all eigenvalues must be zero, there are two cases to be considered, namely the non-degenerate case $(\lambda_1, \lambda_2) = (-\lambda, \lambda)$ and $(\lambda_1, \lambda_2) = (0, 0)$. The former was treated in Koikawa's article and we will focus on the latter, which must correspond to a defective set-up with $\lambda_\star = 0$ as otherwise we would be dealing with a trivial gauge field.

The only independent non-trivial conserved charge reads

$$\mathcal{C}_{(2)} = \frac{1}{2}\left(\psi_1^2 - f_1^2\right) \,, \tag{3.18}$$

which is more than enough to deduce that coloured solutions are such that $\psi_1 = \pm f_1$, from which the solution can be constructed immediately using eqs. (2.11). For illustrative purposes, however, we will outline Koikawa's construction anyway.

Since we are in the $k = 2$ case we have that $\mathsf{L}_{12} = \frac{i}{2} f_1 = -\mathsf{M}_{12}$. Then, the projection of eq. (3.11) along $\vec{e}_1$ can be rewritten as

$$\dot{w}_1(m) = -\mathsf{L}_{12} w_2(m) \,. \tag{3.19}$$

The construction equation (3.9) gives, for the r.h.s. of the above equations

$$\begin{aligned} \mathsf{L}_{12} w_2(1) &= -L_{11}\omega_1(1) + \omega_1(2) \,, \\ \mathsf{L}_{12} w_2(2) &= -L_{11}\omega_1(2) \,, \end{aligned} \tag{3.20}$$

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and eq. (3.16) gives

$$\mathsf{L}_{11} = (\omega_1(2))^2 \,. \tag{3.21}$$

Then, introducing the abbreviations $x \equiv \omega_1(2)$ and $y = \omega_1(1)$, eqs. (3.19) take the form

$$\dot{x} = x^3 \,, \tag{3.22}$$

$$\dot{y} = x^2 y - x \,, \tag{3.23}$$

whose solution is

$$x^{-2} = -2(r+b) \quad , \quad y = -(r+a)\; x(r) \,, \tag{3.24}$$

where a and b are integration constants.

In this simple case the resolvent can be straightforwardly calculated to be

$$\mathsf{R}(\mu) = \frac{-1}{2\mu^2 - \mathcal{C}_{(2)}} \begin{pmatrix} 2\mu + \psi_1 & i f_1 \\ i f_1 & 2\mu - \psi_1 \end{pmatrix} \,, \tag{3.25}$$

and for a coloured solution we must have

$$\mathsf{R}_{11}(\mu) = -\frac{1}{\mu} \; - \; \frac{\psi_1}{2\mu^2} \,. \tag{3.26}$$

On the other hand, from the spectral representation of the resolvent in eq. (3.17) we get

$$\mathsf{R}_{11}(\mu) = -\frac{2xy}{\mu} - \frac{x^2}{\mu^2} \,. \tag{3.27}$$

Comparing both expressions we first of all see that

$$1 = 2xy = \frac{r+a}{r+b} \,, \tag{3.28}$$

whence $b = a$. Secondly we see that $\psi_1 = 2x^2 = -(r+a)^{-1}$, which can be used to deduce f_1 from the constraint $\mathcal{C}_{(2)} = 0$.

The final ingredient in the construction is the imposition of the relevant regularity conditions. In this case we just have to require the Higgs field to be regular on the interval $r \in (0, \infty)$, which implies that $a \geq 0$. In order to make contact with the solution in refs. [2, 7] we redefine $a = \lambda^{-1}$ and find

$$\psi_1 = -\frac{\lambda}{1+\lambda r} = \pm f_1 \quad \xrightarrow{\text{which by eq. (2.8) implies}} \quad \phi_1 = \frac{1}{r(1+\lambda r)} \,. \tag{3.29}$$

This solution corresponds to the coloured solution found by Protogenov in ref. [27] and which was used in refs. [2, 7, 8] to construct coloured black hole solutions to various $\mathcal{N} = 2$ EYM theories; the parameter λ is a hair parameter that doesn't show up in the coloured black hole's asymptotic nor near-horizon behaviour.

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3.2 SU(3)'s coloured solutions

In the SU(3) case there are two coloured possibilities: a pure 3×3 Jordan block or a 2×2 Jordan block; this last case will be briefly treated at the end of this section, and will not lead to a coloured solution.

The explicit forms for the conserved charges are

$$\mathcal{C}_{(2)} = \frac{1}{2}\left(\psi_1^2 - \psi_1\psi_2 + \psi_2^2 - f_1^2 - f_2^2 \right) , \tag{3.30}$$

$$\mathcal{C}_{(3)} = \frac{3}{8}\left(\psi_1 f_2^2 - \psi_2 f_1^2 + \psi_2\psi_1^2 - \psi_1\psi_2^2 \right) , \tag{3.31}$$

from which, unlike the SU(2) case, the coloured solutions are not too easy to deduce.

In the case that L corresponds to a 3×3 Jordan block (so $\lambda_\star = 0$) its spectral representation reads

$$\mathsf{L} = \vec{w}(3)\vec{w}(2)^T + \vec{w}(2)\vec{w}(3)^T \quad \text{whence} \quad \frac{1}{2}\psi_1 = \mathsf{L}_{11} = 2xy , \tag{3.32}$$

where we defined $x \equiv \vec{e}_1 \cdot \vec{w}(3)$, $y \equiv \vec{e}_1 \cdot \vec{w}(2)$ and $z \equiv \vec{e}_1 \cdot \vec{w}(1)$.

Making use of the same technique as in the foregoing section, we find the system of equations

$$\dot{x} = 2x^2 y , \tag{3.33}$$

$$\dot{y} = 2y^2 x - x , \tag{3.34}$$

$$\dot{z} = 2xyz - y . \tag{3.35}$$

This system can be solved by the Ansatz $y = f(r)x\,,\ \ z = g(r)x$. Upon this substitution, the y-equation implies that $f(r) = -(r+a)$ and the z-equation then implies that $\dot{g} = -f$ whence $g(r) = \frac{1}{2}[(r+a)^2 + b]$. The x-equation then implies that

$$\frac{1}{x^2} = 2\left[(r+a)^2 + c\right] , \tag{3.36}$$

from which we see that

$$\psi_1 = -2\frac{r+a}{(r+a)^2 + c} . \tag{3.37}$$

In order to find the full solution we compute the 11-component of the resolvent in eq. (3.17), which reads

$$\mathsf{R}_{11} = -\frac{2xz + y^2}{\mu} - \frac{2xy}{\mu^2} - \frac{x^2}{\mu^3} . \tag{3.38}$$

Calculating the same component using Cramer's rule we see that[10]

$$\mathsf{R}_{11}(\mu) = -\frac{1}{\mu} - \frac{\psi_1}{2\mu^2} - \frac{\psi_1\psi_2 + f_2^2 - \psi_2^2}{4\mu^3} . \tag{3.39}$$

[10]This can be easily derived using

$$\det(\mathsf{L} - \mu) = -\mu^3 + \frac{1}{2}\mu\,\mathcal{C}_{(2)} + \frac{1}{3}\,\mathcal{C}_{(3)} ,$$

and the fact that in the coloured case we must have $\mathcal{C}_{(k)} = 0$.

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Comparing the two expressions we see that we must have

$$1 = 2xz + y^2 \,, \tag{3.40}$$

$$\psi_1 = 4xy \,, \tag{3.41}$$

$$4x^2 = \psi_1\psi_2 + f_2^2 - \psi_2^2 \,. \tag{3.42}$$

Eq. (3.41) was already obtained in eq. (3.37); eq. (3.40) leads to

$$1 = \frac{2(r+a)^2 + b}{2(r+a)^2 + 2c} \quad \text{whence} \quad b = 2c \,. \tag{3.43}$$

The l.h.s. of eq. (3.42) appears in eq. (3.30). Taking into account that in the coloured case $\mathcal{C}_{(2)} = 0$, the solution must be such that

$$f_1^2 = \psi_1^2 - 4x^2 = 2\frac{(r+a)^2 - c}{[(r+a)^2 + c]^2} \,. \tag{3.44}$$

Eliminating from the condition $\mathcal{C}_{(3)} = 0$ the f_2 contribution by means of eq. (3.42) we find

$$\psi_2 f_1^2 = 4x^2\psi_1 \quad \xrightarrow{\text{whence}} \quad \psi_2 = -2\frac{r+a}{(r+a)^2 - c} \,. \tag{3.45}$$

At this point we know ψ_1 and ψ_2 so we can use eq. (3.42) to find f_2^2:

$$f_2^2 = 2\frac{(r+a)^2 + c}{[(r+a)^2 - c]^2} \,. \tag{3.46}$$

If we now introduce the redefinitions $a = \lambda^{-1}$ and $c = \lambda^{-2}\xi$ then we find

$$\begin{aligned} \psi_1 &= -2\lambda\,\frac{1+\lambda r}{(1+\lambda r)^2 + \xi}\,, \qquad & \psi_2 &= -2\lambda\,\frac{1+\lambda r}{(1+\lambda r)^2 - \xi}\,, \\ f_1^2 &= 2\lambda^2\,\frac{(1+\lambda r)^2 - \xi}{[(1+\lambda r)^2 + \xi]^2}\,, \qquad & f_2^2 &= 2\lambda^2\,\frac{(1+\lambda r)^2 + \xi}{[(1+\lambda r)^2 - \xi]^2}\,. \end{aligned} \tag{3.47}$$

For the solution to be well-defined on the range $r \in (0, \infty)$ we must have $\lambda > 0$ and $|\xi| \leq 1$.

The Higgs field is then readily seen to be

$$\phi_1 = \frac{2\,(1+\xi+\lambda r)}{r\,[(1+\lambda r)^2 + \xi]}\,, \qquad \phi_2 = \frac{2\,(1-\xi+\lambda r)}{r\,[(1+\lambda r)^2 - \xi]}\,. \tag{3.48}$$

A special case is given by the solution with $\xi = 0$: in that case the Higgs field is given by

$$\Phi = \frac{1}{r(1+\lambda r)}\,\mathrm{diag}(1, 0, -1)\,, \tag{3.49}$$

from which it is paramount that we are dealing with the embedding SU(2)'s coloured solution in eq. (3.29), into SU(3) by means of the maximal/singular embedding of SU(2) into SU(3) [28].

Figure 1 displays the behaviour of the above family of solutions for the variables $r\phi_i$, which, if we take into account eqs. (2.16) and (2.20), together with the fact that the

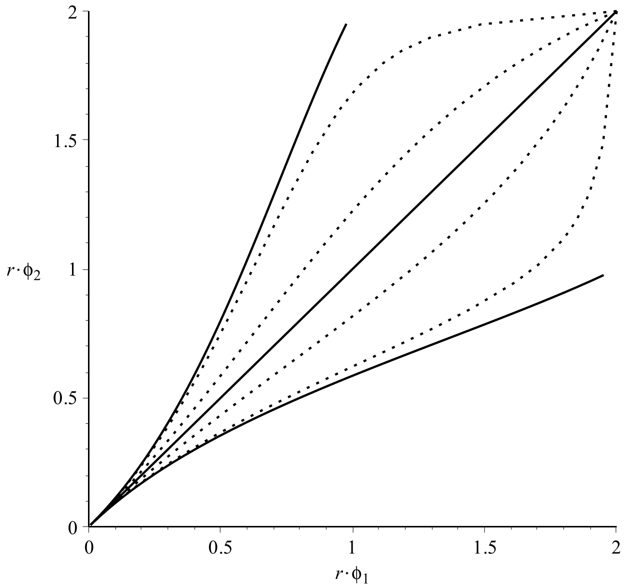

Figure 1. The flow described by the solutions in this section on the $(r\phi_1, r\phi_2)$-plane. The solid line in the center corresponds to $\xi = 0$, the upper solid line to $\xi = -1$ and the lower solid line to $\xi = 1$. The dashed lines correspond to various values of $|\xi| \neq 0, 1$. All flows go towards the origin.

solutions have no Higgs v.e.v., can be thought of as the evolution of the colour charge along the flow parametrized by r.

As promised at the beginning of this section, we now turn our attention to the case in which there is a 2×2 Jordan block in the spectral representation of L. We can choose vectors $\vec{v}$, $\vec{w}(2)$ and $\vec{w}(1)$ such that $\mathsf{L} = \vec{w}(2)\vec{w}(2)^T$ and therefore, since the eigenvalues vanish in the defective case,

$$\mathsf{R}(\mu) = -\frac{1}{\mu}\left[\vec{v}\vec{v}^T + \vec{w}(2)\vec{w}(1)^T + \vec{w}(1)\vec{w}(2)^T\right] \; - \; \frac{1}{\mu^2}\,\vec{w}(2)\vec{w}(2)^T\,. \tag{3.50}$$

The generic expression of the resolvent in eq. (3.39), however, has a μ^{-3} term whereas the chosen spectral representation does not: therefore the coefficient of the μ^{-3} term must vanish, whence $\psi_1\psi_2 + f_2^2 - \psi_2^2 = 0$. This, together with eq. (3.30), implies that $\psi_1^2 = f_1^2$. Eq. (3.31) then implies that $\psi_1(\psi_2^2 - f_2^2) = 0$. The conclusion then must be that either $\psi_1 = f_1 = 0$ and $\psi_2 = \pm f_2$ or that $\psi_2 = f_2 = 0$ and $\psi_1 = \pm f_1$; the non-zero functions are the same as the SU(2) coloured solution in eq. (3.29).

That these solutions do not correspond to the regular embedding of SU(2) into SU(3) becomes clear when we calculate the Higgs field for e.g. the case $\psi_1 = f_1 = 0$:

$$2\Phi = \frac{1}{r}\,\mathrm{diag}(2,-1,-1) + \frac{1}{r(1+\lambda r)}\,\mathrm{diag}(0,1,-1)\,. \tag{3.51}$$

This solution can be thought of as an SU(3) Wu-Yang monopole coinciding with a coloured solution, a combination which is, however, not a coloured solution even though it has no

Higgs v.e.v. It does, however, show the transmutation of colour charge between $r = 0$ and $r = \infty$[11] typical of a coloured solution.

4 $\mathcal{N} = 2, d = 4$ (super-)EYM coloured black holes

The characterisation of supersymmetric solutions to $\mathcal{N} = 2, d = 4$ supergravity coupled to m vector-multiplets with gauged isometries of ref. [2] called $\mathcal{N} = 2, d = 4$ (Super-)EYM does not deal with the most general of such theories [4, 29]. In what follows it is important to understand the restrictions; we will briefly outline the main ingredients and restrictions, referring the reader to ref. [2] for a more detailed discussion.

The bosonic field content of $\mathcal{N} = 2, d = 4$ supergravity coupled to m vector multiplets consists of a metric, $m + 1$ vector fields A^Λ $(\Lambda = 0, \ldots, m)$ and m complex scalar fields $\mathcal{Z}^i$ $(= 1, \ldots, m)$. The scalar fields parametrise a (special-)Kähler manifold $\mathcal{M}$ with a Kähler metric $\mathcal{G}_{i\bar{\imath}} = \partial_i \partial_{\bar{\imath}} \mathcal{K}$, where $\mathcal{K}$ is the Kähler potential. The basic premise of $\mathcal{N} = 2$ EYM is that the metric $\mathcal{G}$ allows for Killing vectors and that we want to gauge a (necessarily non-Abelian!) subgroup $\mathrm{G} \subseteq \mathrm{Isom}(\mathcal{G})$ whose associated Killing vectors, denoted by K_Λ, satisfy

$$[\, K_\Lambda\, ,\ K_\Sigma\,] = -f_{\Lambda\Sigma}{}^{\Omega}\, K_\Omega\ , \tag{4.1}$$

where the f's are the structure constants of the Lie algebra $\mathfrak{g}$ associated to G. Observe that the way we are writing the Killing vectors would mean that we are using $m+1$ isometries: this is the maximal possibility and we can consider other possibilities by taking some K_Λ to vanish.

The couplings of the scalars to themselves (that is, the Kähler metric) and to the other fields are related and constrained by $\mathcal{N} = 2$ supersymmetry and can be described in a unified way by a structure called Special Geometry, see e.g. [3, 4, 30]. The central object in this description is a symplectic section $\mathcal{V}$ belonging to a flat $2(m+1)$-dimensional bundle $E \times L^1 \to \mathcal{M}$ with structure group $\mathrm{Sp}(m+1; \mathbb{R}) \times \mathrm{U}(1)$, that has to satisfy

$$i = \langle\, \overline{\mathcal{V}} \mid \mathcal{V}\, \rangle\ , \tag{4.2}$$

$$0 = \mathcal{D}_{\bar{\imath}} \mathcal{V}\ , \tag{4.3}$$

$$0 = \langle\, \mathcal{D}_i \mathcal{V} \mid \mathcal{V}\, \rangle\ , \tag{4.4}$$

where the bracket denotes the $\mathrm{Sp}(m+1; \mathbb{R})$-invariant innerproduct and the $\mathcal{D}$ denote U(1)-covariant (anti-)holomorphic derivatives. The action of the group we want to gauge can then be lifted to the bundle and leads to the requirement that the symplectic section $\mathcal{V}$ be invariant under G up to $\mathrm{Sp}(m+1; \mathbb{R})$ and U(1) transformations. The restriction imposed in ref. [2] is then the statement that the compensating $\mathrm{Sp}(m+1; \mathbb{R})$ transformations must be such that under the branching to G only singlets and the adjoint representation appears.

The restriction is easier to understand in those situations where the so-called prepotential $\mathcal{F}$ exists [31]: in that case we have a homogeneous function $\mathcal{F}(\mathcal{X})$ of degree 2 such

[11] In other words: the colour charge at spatial infinity and on the horizon are different and the solution smoothly interpolates between these two configurations.

that

$$\Omega = e^{-\mathcal{K}/2}\mathcal{V} = \begin{pmatrix} \mathcal{X}^\Lambda \\ \partial_\Lambda \mathcal{F} \end{pmatrix} \quad \longrightarrow \quad \begin{cases} ie^{-\mathcal{K}} &= \langle \overline{\Omega} | \Omega \rangle \ , \\ 0 &= \partial_{\bar{\imath}} \Omega \ , \\ 0 &= \langle \partial_i \Omega | \Omega \rangle \ , \end{cases} \tag{4.5}$$

where $\mathcal{K}$ is the Kähler potential. Ref. [2]'s restriction then means that the $\mathcal{X}$ transform under G as the adjoint representation and singlets; this action must furthermore be such that

$$0 = f_{\Lambda\Sigma}{}^{\Omega}\ \mathcal{X}^\Sigma\ \partial_\Omega\ \mathcal{F}\ , \tag{4.6}$$

which is the same thing as saying that the prepotential $\mathcal{F}$ is a G-invariant function [31].

The action of the bosonic sector of $\mathcal{N}=2$ super-EYM then reads (see e.g. [4, 29])

$$\begin{aligned} S = \int d^4x \sqrt{|g|}\ \big[R(g) + 2\mathcal{G}_{i\bar{\jmath}} \mathsf{D}_\mu \mathcal{Z}^i \mathsf{D}^\mu \bar{\mathcal{Z}}^{\bar{\jmath}} + 2\mathrm{Im}\,(\mathcal{N})_{\Lambda\Sigma}\ \mathsf{F}^{\Lambda\,\mu\nu}\ \mathsf{F}^{\Sigma}{}_{\mu\nu} \\ -2\mathrm{Re}\,(\mathcal{N})_{\Lambda\Sigma}\, \mathsf{F}^{\Lambda\,\mu\nu}\ \star\mathsf{F}^{\Sigma}{}_{\mu\nu} - V(Z, Z^*) \big]\ , \end{aligned} \tag{4.7}$$

where the gauge-covariant derivative and field strengths are defined as

$$\mathsf{D}\mathcal{Z}^i = d\mathcal{Z}^i + \mathsf{A}^\Lambda\ K_\Lambda{}^i \quad \text{and} \quad \mathsf{F}^\Lambda = d\mathsf{A}^\Lambda - \frac{1}{2}\ f_{\Sigma\Gamma}{}^{\Lambda}\ \mathsf{A}^\Sigma \wedge \mathsf{A}^\Gamma\ . \tag{4.8}$$

$\mathcal{N}_{\Lambda\Sigma}$ is the, model dependent, period matrix and $V(Z, Z^*)$ is the scalar potential[12]

$$V(\mathcal{Z}, \bar{\mathcal{Z}}) \ = \ -\frac{1}{4}\ \mathrm{Im}\,(\mathcal{N})^{\Lambda\Sigma}\ \ \mathcal{P}_\Lambda \mathcal{P}_\Sigma\,. \tag{4.9}$$

In the above equations, $K_\Lambda{}^i(\mathcal{Z})$ are the holomorphic Killing vectors of the isometries that have been gauged and $\mathcal{P}_\Lambda(\mathcal{Z}, \bar{\mathcal{Z}})$ are the corresponding momentum maps; they are related by

$$K^i_\Lambda = i\ \mathcal{G}^{i\bar{\jmath}}\ \partial_{\bar{\jmath}} \mathcal{P}_\Lambda\ . \tag{4.10}$$

Once we have defined the model, one sees that the supersymmetric solutions in the so-called timelike case which is the one the black holes belong to, is constructed out of $2(m+1)$ real seed functions denoted by $\mathcal{I}^\Lambda$ and $\mathcal{I}_\Lambda$. These seed functions are such that if the Λ index corresponds to a singlet under the G-action, the corresponding $\mathcal{I}^\Lambda$ and $\mathcal{I}_\Lambda$ are harmonic functions on $\mathbb{R}^3$, whereas if it corresponds to the adjoint, the Higgs field defined by

$$\mathcal{I}^\Lambda \ = \ -\sqrt{2}\Phi^\Lambda\ , \tag{4.11}$$

must solve the Bogomol'nyi equation (1.1). The corresponding $\mathcal{I}_\Lambda$ must solve an involved equation, see [2, eq. (4.27)], and we will avoid this hurdle by taking them to vanish identically, which is always possible. Observe, however, that this means that the non-Abelian gauge fields will have no electric components and will be purely magnetic.

From this point onwards, the construction of the solution is basically the same as for an ungauged Abelian theory [5, 6]: as we are interested in spherically symmetric, static

[12] Observe that since the imaginary part of the period matrix is negative definite, the scalar potential is positive semidefinite.

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spacetimes we must impose the condition that $\langle\mathcal{I}|\mathsf{D}\mathcal{I}\rangle = 0$, where D is the G-covariant derivative. Having satisfied said condition, the metric becomes

$$ds^2 = e^{2U}dt^2 - e^{-2U}\, d\vec{x}^2_{(3)} \qquad \text{where} \qquad e^{-2U} = \mathrm{W}(\mathcal{I})\,, \tag{4.12}$$

is a model-dependent homogeneous function of degree 2 called the Hesse potential [32–34]. The scalar fields are model-dependent but they are nicely expressed in terms of the Hesse potential as

$$Z^i = \frac{\tilde{\mathcal{I}}^i + i\mathcal{I}^i}{\tilde{\mathcal{I}}^0 + i\mathcal{I}^0} \qquad\qquad \text{where} \qquad \tilde{\mathcal{I}}^\Lambda = -\frac{1}{2}\,\frac{\partial \mathrm{W}}{\partial \mathcal{I}_\Lambda}\,. \tag{4.13}$$

The final step in the construction consists in adjusting the integration constants of the seed functions so that the solution describes the geometry of an extremal black hole outside its horizon: in order for a spherically symmetric metric of the type in eq. (4.12) to be interpreted as a black hole, it should be asymptotically flat and singular at $r = 0$ only and this singularity must be such that the limiting geometry at $r = 0$ is that of an $aDS_2 \times S^2$ spacetime. With these provisos, the area of the 2-sphere in this limiting geometry corresponds to the area of the horizon of the black hole and therefore also to the entropy of the black hole (See e.g. [35]). For a given Hesse potential and given seed functions, this entropy can be calculated straightforwardly but need not be finite or positive: the regularity of the metric at the horizon is then, by means of the entropy, a constraint on the parameters of the solution.

Further constraints on the parameters of the seed functions come from the fact that the metrical factor e^{-2U} must never vanish on the interval $r \in (0, \infty)$, as otherwise the solution would have a curvature singularity on said interval, whence ruining the interpretation of the solution as describing the exterior of an extremal black hole.

Having outlined the restrictions on the theories and the solutions, we are ready to use the coloured solutions to the SU(3) Bogomol'nyi equations obtained in section 3.2 to build supersymmetric coloured black holes.

4.1 $\overline{\mathbb{CP}}^{\mathbf{8}}$ coloured black holes

The so-called $\overline{\mathbb{CP}}^8$ model has 9 vector fields, a scalar manifold that is the symmetric space $\mathrm{SU}(1,8)/\mathrm{U}(8)$ and is defined by the prepotential

$$\mathcal{F} = \frac{1}{4i}\,\eta_{\Lambda\Sigma}\,\mathcal{X}^\Lambda\mathcal{X}^\Sigma \qquad\qquad \text{where}\;\; \eta \;=\; \mathrm{diag}(+,[-]^8)\,, \tag{4.14}$$

and the indices Λ and Σ run from 0 to 8. Since the prepotential is manifestly $\mathrm{SO}(1,8)$ invariant and we have 9 vector fields at our disposal we can at most gauge a 9-dimensional subgroup $\mathrm{G} \subset \mathrm{SO}(1,8)$; if we couple this to the restriction that in the branching of $\mathrm{SO}(1,8)$'s $\mathbf{9}$ we must only find the adjoint and singlets of G, then we see that the singular embedding of SU(3) into SO(8) does the trick [28].

Sticking to a purely magnetic solution, so that $\mathcal{I}_0 = 0$, we see that the metrical factor becomes

$$e^{-2U} = \frac{1}{2}\left(I^0\right)^2 - \frac{1}{2}I^aI^a = \frac{1}{2}\left(I^0\right)^2 - 2\mathrm{Tr}\left(\Phi^2\right) = H^2 - \phi_1^2 + \phi_1\phi_2 - \phi_2^2\,, \tag{4.15}$$

where in the last step we redefined $I^0 = \sqrt{2}H$. Since H is a SU(3)-singlet, it has to be a harmonic function and, to preserve spherical symmetry, it has to be of the form

$$H = h + \frac{p}{r} \qquad \text{so that} \qquad \lim_{r\to\infty} e^{2U} = h^2 + \frac{2hp}{r} + \ldots \tag{4.16}$$

where we have already used the asymptotic vanishing of the Higgs field characteristic of coloured solutions. We take $h^2 = 1$ in order for the metric to become asymptotically Minkowski in spherical coordinates. The mass is then given by $M = hp$, which can always be taken to be positive by taking $h = \text{sign}(p)$, whence $M = |p|$. The entropy of the non-Abelian black hole is readily calculated to be

$$\frac{S}{\pi} = \begin{cases} p^2 - 4 : & \xi \neq \pm 1 \\ \\ p^2 - 3 : & \xi = \pm 1 \end{cases} \tag{4.17}$$

where the 4 and the 3 are the monopole's contribution to the entropy. We must have $|p| > 2$ when $\xi \neq 1$ and $|p| > \sqrt{3}$ when $\xi = \pm 1$ in order for the entropy to be finite and positive. As in the ungauged case, positive mass and a well-defined entropy of the horizon is, at least for these solutions, enough to ensure the regularity of the metric.

The eight complex physical scalars in this theory, $\mathcal{Z}^a$ $(a = 1, \ldots, 8)$ are given by

$$\mathcal{Z}^a = \frac{I^a}{I^0} = -\frac{\Phi^a}{H} \xrightarrow{\text{SU(3) defining rep.}} \mathcal{Z} = -\frac{\Phi}{H}\,. \tag{4.18}$$

In the last expression it is paramount that the scalars behave asymptotically as $\mathcal{Z} \sim \mathcal{O}(r^{-2})$ and that near the horizon they behave as

$$\lim_{r\to 0} \mathcal{Z} = \begin{cases} -\frac{1}{p}\,\text{diag}(1, 0, -1) : & \xi \neq \pm 1 \\ \\ -\frac{1}{2p}\,\text{diag}(2, -1, -1) : & \xi = +1 \\ \\ -\frac{1}{2p}\,\text{diag}(1, 1, -2) : & \xi = -1 \end{cases} \tag{4.19}$$

Let us stress once again that even though the near-horizon behaviour is determined by (colour) charges only the Abelian charge p is asymptotically measurable. Also observe that the hair parameters λ and ξ do not influence the asymptotic nor the near-horizon behaviour, illustrating once again the non-applicability of the no-hair theorem in gravity coupled to YM theories.

4.2 $\mathbb{C}$-magic coloured black holes

The $\mathbb{C}$-magic model is a model with 10 vector fields and 9 complex scalars parametrising the symmetric space $\text{SU}(3,3)/S\,[\text{U}(3) \times \text{U}(3)]$. A convenient prepotential for this model was given in ref. [36] and splits the 10 complex coordinates $\mathcal{X}^\Lambda$ into a singlet under the

$\mathrm{SU}(3)\times\mathrm{SU}(3)$ action, $\mathcal{X}^0$, and a 3×3 matrix that we denote by $\mathcal{X}$, transforming as the $(\mathbf{3},\bar{\mathbf{3}})$ irrep. The prepotential can then be expressed as

$$\mathcal{F} \;=\; \frac{\det(\mathcal{X})}{\mathcal{X}^0}\,. \tag{4.20}$$

As was argued in ref. [2] this model allows for a gauging of the diagonal SU(3), which seeing the branching rule $(\mathbf{3},\bar{\mathbf{3}})\to\mathbf{1}\oplus\mathbf{8}$ [28] is exactly what is needed to fulfill the constraints outlined in section 4.

Ref. [2] then used the spherically symmetric Wilkinson-Bais monopoles [24] for SU(3) to construct supersymmetric globally regular solutions describing the (super-)gravitational backreaction of said monopoles and also used a small generalisation to construct supersymmetric non-Abelian black hole. All those solutions have, however, asymptotic colour charge and non-vanishing Higgs v.e.v. and in this section we are going to use the solutions of the Bogomol'nyi equation derived in section 3.2 to construct coloured black-hole solutions to the $\mathbb{C}$-magic model.

Following ref [2], we shall take $\mathcal{I}^0=0$, which means that the Abelian gauge field A^0 is purely electric, and also $\mathcal{I}_{\Lambda\neq 0}=0$. This last choice leads to a static solution with a metric factor with the simple form

$$e^{-2U} \;=\; \sqrt{H\ \det(K\ \mathrm{Id}-2\Phi)} \;=\; \sqrt{H\ (K-\phi_1)(K-\phi_2+\phi_1)(K+\phi_2)}\ , \tag{4.21}$$

where we defined $\mathcal{I}_0=\frac{1}{4\sqrt{2}}\,H$ and took the singlet seed function to be $\frac{1}{\sqrt{2}}K$.[13] As H and K are singlets under the gauge group, they are harmonic functions and taking them to be spherically symmetric they can be expanded as

$$H=h+\frac{q}{r}\quad,\quad K=k+\frac{p}{r}\,. \tag{4.22}$$

The criterion for the absence of coordinate singularities for any $r\in(0,\infty)$ immediately implies that $\mathrm{sign}(h)=\mathrm{sign}(q)$.

By considering the asymptotic behaviour of the metrical factor in eq. (4.21) we can straightforwardly normalise the solution to asymptote to ordinary Minkowski spacetime by taking $hk^3=1$ which then leads to the following expression for the mass

$$M=\frac{k^3q+3k^{-1}p}{4}\,. \tag{4.23}$$

k is related to the values of the scalars at spatial infinity. The actual expression is complicated and not very enlightening.

As one can see from figure 1 the limit $r\to 0$ differs substantially between the cases $\xi\neq\pm1$ and $\xi=\pm1$ and we will discuss the regularity properties of the solution seperately. From the metrical factor in eq. (4.23) it follows that the case $\xi=-1$ can be obtained from the case $\xi=1$ by substituting $\phi_1\leftrightarrow\phi_2$, $K\to -K$ and $H\to -H$, and, accordingly, we shall not discuss the construction of a coloured black-hole solution for the case $\xi=-1$.

[13] Observe that ref. [2] uses the definition $\mathcal{I}_0=\frac{1}{\sqrt{2}}\,H$, which means that the metrical factor in [2, eq. (5.47)] is missing a factor of two. In said equation, the harmonic function K is denoted by λ, a naming we changed here in order to avoid confusion.

4.2.1 $\xi \neq \pm 1$

In this case the entropy reads

$$S(\xi \neq \pm 1) = \pi \sqrt{qp(p^2-4)} \ . \tag{4.24}$$

there are four possible conditions on the charges that make the entropy well defined and finite:

Case a) $p > 2$ and $q > 0$. The absence of zeroes in the function H then implies that $h > 0$, whence also $k > 0$ by the normalisation condition; the mass as calculated by eq. (4.23) is automatically positive.

Case b) $p < -2$ and $q < 0$. This further implies $h < 0$ and $k < 0$ implying that the mass is automatically positive.

Case c) $p \in (0, 2)$ and $q < 0$. Seeing that we must have $h < 0$ and $k < 0$, the mass is not automatically positive. One can, however, see fairly rapidly that in this case the combination $H - \phi_2$ appearing in the metrical factor (4.21) has a zero on the interval $(0, \infty)$: conforming to the criteria outlined above, this means that case c) is not a viable option and must be discarded.

Case d) $p \in (-2, 0)$ and $q > 0$. This case must be supplemented by the conditions $h > 0$ and $k > 0$, which then means that the factor $H - \phi_1$ has a zero, whence this case must also be discarded.

For the cases a) and b) one can see that the resulting metrical factor has no zeroes for $r > 0$ and the corresponding solutions describe coloured black holes.

4.2.2 $\xi = 1$

The entropy reads

$$S(\xi = 1) = \pi \sqrt{q(p+1)^2(p-2)} \ . \tag{4.25}$$

There are possibilities:

Case α) $p > 2$ and $q > 0$, which following the same reasoning as before leads to the further constraints $h > 0$ and $k > 0$ and therefore also to positive mass.

Case β) $p < -1$ and $q < 0$, whence also $h < 0$ and $k < 0$, which also implies that the mass is positive.

In these two cases the metrical factor is free of coordinate singularities on the interval $(0, \infty)$ and do define coloured black holes.

Observe that as far as the entropy is concerned, we could allow for the possibility of $p \in (-1, 2)$. This possibility requires $q < 0$, $h < 0$ and $k < 0$, which automatically implies that the factor $H + \phi_2$ in the metrical factor has a zero and must therefore be discarded.

5 Conclusions

In this article we have constructed coloured black hole solutions to two models of $\mathcal{N} = 2, d = 4$ EYM theories with an SU(3) gauge group, namely the $\overline{\mathbb{CP}}^8$ and the $\mathbb{C}$-magic model. This construction was possible due to the fact that the Bogomol'nyi equation, a prominent ingredient in the construction of supersymmetric solutions to the used class of theories, under the assumption of (maximal) spherical symmetry is an integrable system. The system admits a Lax pair and after having identified the solutions needed to construct coloured black holes as corresponding to defective Lax matrices, these were constructed for the SU(3) Bogomol'nyi equation. The coloured black holes built upon these solutions have the same characteristics as the SU(2) black holes: there is no asymptotic colour charge, one always needs extra active Abelian fields and most importantly of all the horizon is colourful and the entropy depends only on the colour charge on the horizon and not on the hair parameters of the solutions.

The transmutation solution in eq. (3.51) leads to a type of black hole that is unavailable in the SU(2)-gauged models considered up till now: as one can see, the solution is such that the asymptotic colour charge is non-vanishing but different from the horizon colour charge and it can be used to construct black hole solutions along the lines outlined in the last section. Consider for example the case of $\overline{\mathbb{CP}}^8$ treated in section 4.1, the embedding into the $\mathbb{C}$-magic model being also straightforward: the metrical factor can be expanded as

$$e^{-2U} = h^2 + \frac{2hp}{r} + \frac{p^2-4}{r^2} + \frac{2\lambda}{r(1+\lambda r)^2}\,. \tag{5.1}$$

According to the criteria outlined in section 4 we see that in order to have a regular black-hole solution we must have $h = \mathrm{sign}(p)$ so that the mass is $M = |p| \geq 0$ and $|p| > 2$ so that the entropy, given by $S = \pi(p^2 - 4)$ is finite and positive. These two conditions suffice to make the metrical factor a sum of positive terms, producing a perfectly well-defined black-hole solution. By looking at the expression for the scalars in eq. (4.19) one can readily see that they are also well defined and behave near the horizon as the $\xi \neq \pm 1$ case in eq. (4.19).

In section 3.2 we saw that the SU(3) coloured solution is related to a 3×3 Jordan block, whereas the transmutation solution is related to the appearance of a smaller Jordan block. In fact, it can be shown that the SU(N) coloured solutions can only follow from a Lax matrix that is similar to an $N \times N$ Jordan block, meaning that the transmutation solutions are the rule rather the exception for $N > 3$. The question of whether these transmutation solutions can, like the one for SU(3), be written as the sum of Wu-Yang monopoles and coloured solutions for smaller N should be worth investigating.

The coloured and the transmutation black holes have the characteristic that the asymptotic colour charge does not match the colour charge seen on the horizon. Strictly speaking, non-Abelian charges can only be defined globally, at infinity but the calculation of the entropy indicates clearly the presence of non-Abelian charges, different from those seen at spatial infinity, which contribute to it. This is a very intriguing phenomenon which calls for further investigation. A microscopic interpretation of the entropy of non-Abelian black holes (coloured or with asymptotic charges) is badly needed.

As for the celebrated attractor mechanism [37–40], it works (in the covariant sense discovered in ref. [1]), but only in terms of the horizon charges. A further difference from the well-known Abelian case is the fact that the asymptotic value of the scalars related to the Higgs field is not arbitrary: it has to vanish for coloured black holes, as discussed in section 2.

We hope to have convinced the reader that the physics of non-Abelian charged black holes has very interesting features that go far beyond the well-known existence of non-Abelian hair and deserves further investigation. Work in this direction is in progress.

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